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W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

EDITED BY

KARL PEARSON

ASSISTED BY

EGON S. PEARSON

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VOLUME XXIII DECEMBER, 1931 PARTS III AND IV

BIOMETRIKA

A CONTRIBUTION TO THE CRANIOLOGY OF THE EASTER ISLANDERS.

BY GERHARDT VON BONIN, DR.MED. From the Department of Anatomy, University of Illinois.

THE object of this paper is to put on record a description of the skulls of the Easter Islanders, and to give some indications as to the place of this race within the human stock, postponing a more comprehensive survey of the inhabitants of the South Sea until more material is available.

By good fortune, the author has been enabled to study skulls from Easter Island—or, as it is often but apparently not quite correctly called, Rapanui—in various collections. The largest number was found in the Hunterian Museum of the Royal College of Surgeons in London. These skulls were almost all collected in 1914–15 by Mrs Routledge, whose book contains valuable information about the history and the ethnology of the island. A second collection was that of the Ethnographical Museum in Leiden. These skulls were purchased in 1885 from a man of the name of J. Weisser*. Eleven more skulls from the British Museum (Natural History) were added to these. They were brought home by the late Lord Crawford who in 1903 had obtained them from a native while in Easter Island (see Nicoll, 1908).

My sincerest thanks are due to all those who gave me permission to use the material under their care, Sir Arthur Keith, Professor Juinboll, and Mr Pycraft.

Work on the present paper was carried on at different times in the following places: the Laboratory of the Royal College of Surgeons of England, the Biometric Laboratory of University College, London, the Departments of Anatomy of the University of Leiden and of the University of Illinois in Chicago. My thanks are due to Professors Sir Arthur Keith, Karl Pearson, J. A. J. Barge and O. F. Kampmeier. I further wish to express my gratitude to those persons who have helped me in other ways; Miss Spencer Wilkinson and Miss S. Levin have taken great labour in setting up the table of measurements, and Miss Marion Mason has very kindly undertaken to draw the accompanying illustrations as well as the type contours.

The present series of Easter Island skulls could be compared with another one previously published by Volz. He had skulls which were collected in 1882 by the

* I am indebted to Professor Juinboll for the following letter indicating the source of the material in the collection in Leiden: "In antwoord op bovenaangehaald schrijven kan ik *U* mededeelen, dat volgens den schedelcatalogus de door *U* bedoelde schedels zijn aangekocht in November 1885 van J. Weisser (vermoedelijk een handelaar)." See Note p. 269 below.

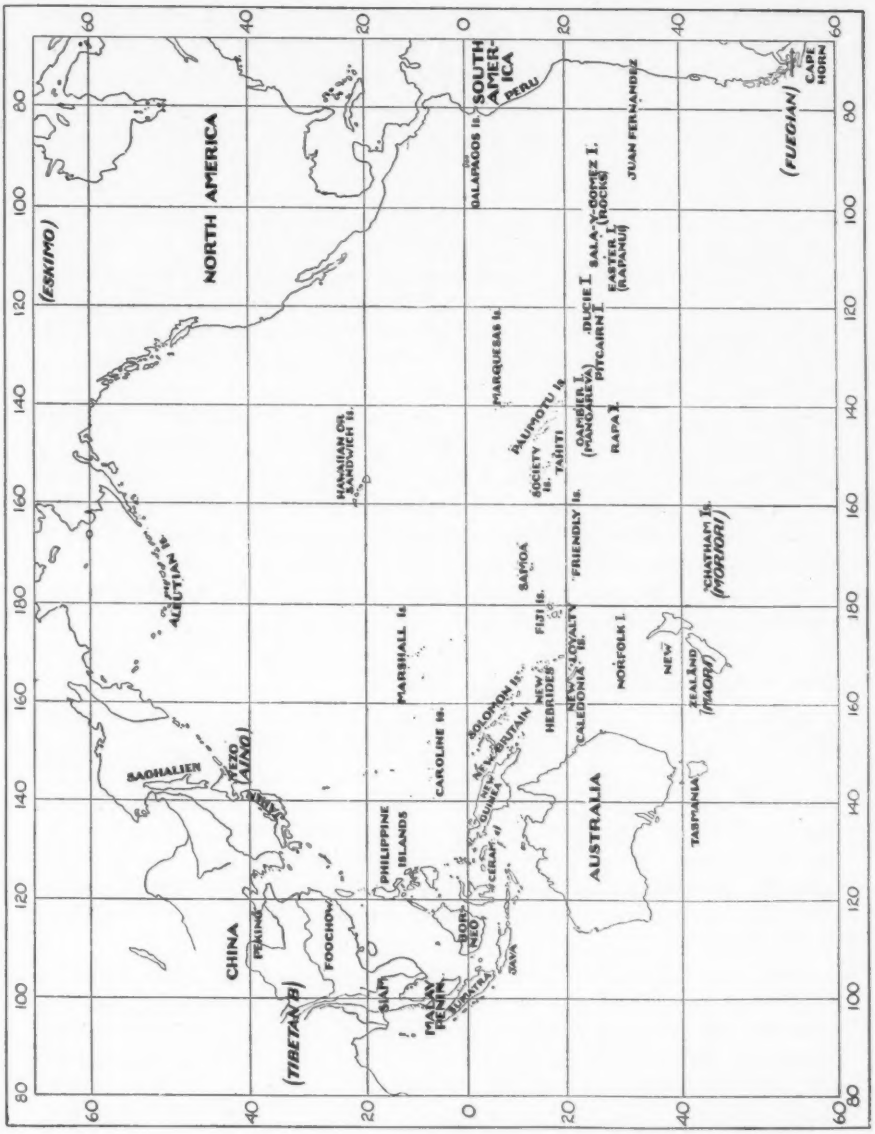
expedition of the *Hyäne*. Part of these skulls had been deposited in Berlin, another part in Dresden. Volz has given measurements of both. Later, Meyer and Jablonowski re-examined the Dresden skulls; their report, an admirably sober piece of work, contains a detailed discussion of the history and prehistory of the island, and gives information as to the technique of measurements which can not be taken from Volz's paper. In regard to a few skulls they differed in the matter of sexing. Meyer and Jablonowski's data have here been accepted.

The name "Easter Island" conjures up problems most of which, however, do not fall within the scope of craniology. Situated as that island is far away from any other in the South Sea, it has always been considered a matter of mystery that natives from other shores ever reached it at all. This sense of mystery has been enhanced by the discovery of a fairly high level of culture. The well-known stone statues with which the island seems to abound and the use of a sort of script have both excited curiosity. Native tradition has it that two sets of settlers reached the island. Ethnologists assert that traits of both Melanesian and Polynesian culture are to be found on the island. The problem whether these two groups came from different racial stocks has claimed the attention of anthropologists who have worked on the inhabitants of this island. "Long Ears" and "Short Ears" seem to have arrived on the island in this chronological order, and seem to have fought each other, until the "Short Ears" definitely ruled the land. The view that the former are of Melanesian and the latter of Polynesian stock is held by Volz and Routledge, but is refuted by Meyer and Jablonowski. What the answer of Biometry is will be seen presently.

It is unnecessary to go into the history of Easter Island; this has been analyzed by Meyer and Jablonowski. Only a few facts are of interest to physical anthropology. Taking together all the accounts of travellers who have visited the island at various times within the last few centuries, there can be little doubt that its population has considerably decreased within that period. The Peruvians raided the island in 1863 and carried off what seems to have been practically the whole population for slave work in the Peruvian mines. Owing to outside pressure part of these unfortunate victims were brought back to their home after a year, but small-pox had been acquired by them in America and took a heavy toll after they had settled down again. It appears further that at some later period when the island was exploited commercially, workmen from Tahiti lived there for some time. These two events—the Peruvian raid and the importation of labour—may have tended to alter the physical type. The skulls to be described here and those which have been described by Volz all seem to have come from natives who were born prior to 1862. Volz's skulls were collected in 1882, those in Leiden prior to 1885. Crawford definitely asserts that his skulls came from the "platforms" of the old stone monuments and the same holds true for those brought home by Mrs Routledge.

The measurements given here are those usually taken in the Biometric Laboratory. The author has already availed himself of another opportunity to express his sincerest thanks to Dr G. M. Morant who very kindly with the

permission of Professor Karl Pearson showed him all the details of the technique. The designation of the measurements, too, follows biometric routine. A few points,



PACIFIC OCEAN IN MERCATOR PROJECTION

however, may deserve a short note. It was impossible, owing to various circumstances, to take the cranial capacity of the skulls in Leiden. At the College of Surgeons no balance was at hand for this purpose, and the capacities had to be ascertained by repacking the seeds into a graduated cylinder. A *crâne étalon* was available which permitted the comparison of the seed capacity with the true water capacity. This was done several times. The values found for seed were:

May 31	1530 c.c.
June 3	1520
4	1515
6	1520
8	1530

The true capacity of this skull as measured by water was 1470 c.c., which according to my data leads to my personal equation 1522·5 seed = 1470 water. This seems to be the same sort of procedure as that followed by Volz whose values agree very closely with those found in London ($\alpha(C) = \cdot 16$). Mustard seed was used, and was packed as tightly as possible.

The callipers employed were purchased from Stanley and from Herrmann, their scales agreeing with each other. Some difficulty was encountered in procuring a reliable tape, steel tapes purchased in London differing to the extent of as much as 0·5 cm. for a length of 50 cm. from the scales of the callipers and engine divided ruler. A tape purchased in Holland, and apparently of French make, was found to be accurate.

When measurements had been completed, it was found that the orbital height (O_2) taken by the author was slightly (somewhat less than 1 mm.) smaller than that measurement taken by Morant or Tildesley. Consequently it was taken again. These second readings are given in the tables.

In London the ear-height was taken on a craniophor constructed by Wingate Todd, in Leiden on an old craniophor in the possession of the Museum. The former had a scale and a rod by which the ear-height could be read off directly, but the latter had to be employed together with a "Stativgoniometer" for taking *OH*. This implies a slight difference in technique: the former *OH* is taken from the ear-rods, the latter exactly from the auricular points. It is believed that even in longer series this difference will not be more than 0·5 mm. at the most, but it appears preferable to use *H′* or *H* for comparative purposes. This has here been done throughout.

Only a few descriptive notes were taken. The states of ossification of sutures were jotted down in some detail, but an attempt at analysis proved most unsatisfactory, for the sequency of closure was found to be rather erratic. They have therefore not been given here. Metopic sutures occurred in skulls 01 and 49. In both cases they did not extend beyond the glabella.

Skulls 15 and 13 showed a canalis basilaris.

An epipteric bone was found in 35 (*R*), 40 (*L*), 50 (*L*), 51 (*R*), and 55 (*R*), a fronto-temporal articulation in 32 (*L*) and 45 (*R*). Tympanic exostoses were present in skulls 8, 10, 11, 19, 29 and 39.

The configuration of the lower border of the pyriform aperture was noted in a number of skulls. Of 26 skulls it was described as sharp in 14 and as blunt in 12. In other words percentages of both were sensibly equal. It is the difficulty in interpreting the anterior border of the nasal floor in cases in which the aperture is blunt that makes the measurement *NH'* appear more exempt from the personal equation of the observer than *NH*, *R* or *L*.

The writer found it possible to take contours only in Leiden. This was on a very small amount of material, but since the Leiden skulls happen to be nearest to the mean values found for all the skulls available for this study, it was thought that an inclusion of these contours would serve a useful purpose. In constructing these type contours, the technique of the Biometric School has again been followed closely*. The only point left out on the sagittal type contours was the Inion. Its definition appeared too ambiguous. The figure in Martin's *Lehrbuch* suggests its position at the upper base of the occipital protuberance, and this would be in accord with Schwalbe's definition. On the other hand, the text of Martin's *Lehrbuch* as well as the discussion of Stefanie Oppenheim place it at the centre of the tuberculum linearum, below the occipital protuberance. In view of these difficulties it was deemed safest to omit this point altogether.

On the other hand, it appeared desirable to introduce into the sagittal contours the Prosthion, which plays such a large rôle in most craniometric papers coming from Germany, and which has also been adopted by other craniologists, among whom Davidson Black may be cited. The present regrettable state of confusion regarding the facial triangle can only be improved upon and reasonable comparison made when both the Alveolar Point and the Prosthion will have been made available for a greater number of series.

Comparison of different series was carried through by the method of the Coefficient of Racial Likeness.

The C.R.L. was used in its abbreviated form:

$$\text{C.R.L.} = \frac{1}{m}\Sigma\alpha - 1 \pm \cdot 67449\sqrt{\frac{2}{m}}.$$

Its standardised value (for $\tilde{n}_1 = \tilde{n}_2 = 100$) was computed by evaluating the expression

$$\text{C.R.L.}' = 50 \times \frac{\tilde{n}_1 + \tilde{n}_2}{\tilde{n}_1 . \tilde{n}_2}\left[\frac{1}{m}\Sigma\alpha - 1 \pm \cdot 67449\sqrt{\frac{2}{m}}\right].$$

When available, the following measurements were used for computation: *C*, *L*, *B*, *B'*, *H'*, *LB*, *S*, *U*, *Q'*, *G'H*, *J*, *NH'*, *NB*, O_1, O_2, G_1, G_2, *fml*, *fmb*, *Pr. P*∠, *A*∠,

[* This is certainly not true in the case of the transverse contour, Fig. 5, where the Biometric School takes *M* as the midpoint of the lowest horizontal and *MA* as the perpendicular bisector. A like serious divergence from biometric methods occurs in the author's diagrams, pp. 97—101 of the present volume. ED.]

N∠, *Oc. I.*, 100 *B*/*L*, 100 *H'*/*L*, 100 *B*/*H'*, 100 *G'H*/*GB*, 100 *NB*/*NH'*, 100 O_2/O_1, 100 G_2/G_1, 100 *fmb*/*fml*. The standard deviations of the Egyptian "E" series were used throughout. Comparisons have been restricted to male series, for they were larger and appeared therefore more reliable.

The author's interpretation of Volz's measurements in most instances becomes clear upon looking at Table I. *NH'* was identified from the paper by Meyer and Jablonowski who state expressly that they measured *NH'*, and whose measurements agree very closely with those given by Volz. Which orbital measurements Volz took remains doubtful, and it was considered safer to leave them out altogether. For there is little room for doubt that the orbit is one of the most dangerous pitfalls in racial comparison! Volz's profile angle was taken to be the Prosthion Angle. His angles of the fundamental facial triangle have been determined by means of drawings

TABLE I.

Means of series of Easter Islanders, all males.

	Volz	R. C. S. London	E. M. Leiden
C	1451·7 (18)	1472·8 (18)	—
L	188·2 (18)	192·3 (36)	189·4 (13)
F	—	190·3 (36)	188·7 (13)
B	135·7 (18)	132·6 (25)	133·4 (13)
H'	143·5 (18)	143·4 (28)	141·5 (12)
H	142·6 (18)	144·9 (29)	142·9 (12)
OH	123·3 (18)	124·2 (31)	122·4 (13)
Br. OH	—	123·4 (31)	121·4 (13)
LB	110·8 (18)	111·2 (29)	109·1 (11)
B'	92·7 (18)	93·8 (32)	92·9 (12)
U	512·1 (17)	519·4 (28)	516·1 (12)
S	372·1 (17)	384·1 (26)	378·0 (12)
Q'	—	326·3 (25)	322·2 (12)
fml	35·3 (17)	36·2 (21)	36·6 (11)
fmb	30·5 (17)	30·2 (23)	31·2 (11)
G'H	69·9 (15)	70·6 (26)	67·6 (11)
J	134·3 (17)	135·3 (21)	133·4 (10)
NH'	52·1 (16)	51·5 (29)	48·9 (12)
NB	27·4 (17)	27·5 (29)	26·8 (12)
O_1R	—	43·4 (27)	42·6 (10)
O_2R	—	34·4 (28)	34·1 (10)
G_1	—	52·3 (21)	48·3 (8)
G_2	—	40·3 (15)	40·4 (3)
P∠	—	88°·2 (26)	90°·8 (11)
Pr. P∠	85°·3 (15)	85°·3 (25)	87°·7 (11)
N∠	68°·4 (14)	66°·5 (23)	64°·2 (11)
A∠	74°·3 (14)	75°·6 (23)	78°·6 (11)
B∠	37°·4 (14)	37°·8 (23)	37°·1 (11)
Oc. I.	—	59·2 (25)	60·3 (13)
100 *B*/*L*	72·2 (18)	69·0 (25)	70·4 (13)
100 *H'*/*L*	76·3 (18)	74·6 (28)	74·7 (12)
100 *B*/*H'*	94·6 (18)	92·4 (23)	94·2 (12)
100 *G'H*/*GB*	70·9 (15)	71·8 (24)	71·0 (10)
100 *NB*/*NH'*	52·8 (16)	53·4 (27)	54·9 (12)
100 O_2/O_1R	—	79·4 (27)	81·3 (10)
100 G_2/G_1	—	77·2 (14)	84·6 (3)
100 *fmb*/*fml*	86·5 (17)	83·5 (20)	85·6 (11)

from the original measurements of the component lengths. The indices $100\,B/L$, $100\,H'/L$, $100\,B/H'$ and $100\,(B-H')/L$ were calculated from his data. It has already been mentioned that Meyer and Jablonowski's sexing for the Easter Island skulls was accepted. The following numbers of Volz's list were then used:

♂: 1785, 151, 154, 147, 1786, 145, 1773, 148, 1784, 143, 137, 156, 139, 1787, 1783, 1767, 133, 142.

♀: 1781, 1776, 1765, 141, 134, 1771, 135, 1779, 138, 1775, 1788, 1774, 136, 1782, 155, 1769, 140.

Juv.: 144, 1772, 153, 146, 1777, 1778, 1770, 150, 1766, 152, 1780, 157.

Volz has given altogether three lengths of the skulls he measured: Gerade Länge, Grösste Länge, and Glabellarlänge. It is not clear to the writer what Volz meant by these terms, the greatest value of the three (the last two were of equal value for quite a number of skulls) was taken as equivalent to the biometric L.

The sexing of the material here put on record has been done by appreciation, since no records were available. There were only a few skulls which gave rise to serious doubts, but a definite decision was made in every case, sometimes after measuring L, B, and H' and assigning "marks" to the skull in question in the manner explained by Morant for the British Neolithic skulls. The means thus arrived at for the two sexes show differences of a magnitude usually found, and hence the sexing, although possibly wrong in some particular instances, can, it is thought, be considered substantially correct.

As has already been stated, the first problem to be answered was whether or not the population could be considered homogeneous. For this purpose, the series from the different institutions were taken separately and compared by means of the C.R.L. The means obtained are given in Table I, and the C.R.L.'s deduced from them in Table II. Only very few measurements showed significant differences ($\alpha > 5$). It will

TABLE II.

C. R. L.'s between groups of Male Easter Islanders.

	Volz	R. C. S.	Leiden
Volz All Characters Indices & Angles	— —	1·40 ± ·19 (23)* 2·13 ± ·30 (9)	1·33 ± ·20 (22)† 2·42 ± ·30 (9)
R. C. S. All Characters Indices & Angles	1·40 ± ·19 (23) 2·13 ± ·30 (9)	— —	·81 ± ·18 (27)‡ ·95 ± ·27 (11)
Leiden All Characters Indices & Angles	1·33 ± ·20 (22) 2·42 ± ·30 (9)	·81 ± ·18 (27) ·95 ± ·27 (11)	— —

* $\alpha > 5$: $\alpha\,(L) = 6{\cdot}11$, $\alpha\,(S) = 9{\cdot}46$, $\alpha\,(100\,B/L) = 14{\cdot}92$.

† $\alpha > 5$: $\alpha\,(NH') = 8{\cdot}24$, $\alpha\,(N\angle) = 9^{\circ}{\cdot}92$, $\alpha\,(A\angle) = 9^{\circ}{\cdot}52$.

‡ $\alpha > 5$: $\alpha\,(NH') = 6{\cdot}73$, $\alpha\,(A\angle) = 5^{\circ}{\cdot}59$.

be observed that only the Leiden, the College of Surgeons and Volz's series were thus compared, the few skulls from the British Museum having been left out, because their small number did not seem to permit the application of this method.

TABLE III.

Means for Easter Island Crania.

A. *Absolute Measurements.*

	Males	Females
C	1462·2 (36)	1304·8 (26)
L	190·6 (72)	179·5 (42)
F	189·7 (54)	180·4 (25)
B	133·3 (60)	129·6 (37)
H'	142·9 (62)	137·3 (40)
H	143·8 (59)	137·9 (33)
OH	123·6 (62)	118·9 (33)
Br. OH	122·8 (44)	—
LB	110·5 (62)	103·1 (40)
B'	93·2 (67)	90·7 (37)
U	516·0 (61)	493·0 (38)
GU	527·1 (42)	499·5 (21)
Q	325·0 (37)	314·0 (15)
Br. Q	319·5 (42)	308·0 (21)
S_1'	115·5 (53)	110·3 (25)
S_2'	114·4 (52)	110·9 (25)
S_3'	100·5 (43)	98·7 (21)
S_1	133·3 (52)	126·3 (42)
S_2	127·5 (51)	121·2 (42)
S_3	121·7 (43)	116·3 (38)
S	378·9 (59)	362·6 (38)
fml	35·9 (53)	34·1 (32)
fmb	30·5 (55)	29·5 (37)
G'H	69·6 (55)	62·9 (30)
J	134·3 (52)	124·3 (28)
GB	98·0 (59)	91·7 (27)
GL	102·5 (54)	98·5 (31)
PH	17·8 (28)	15·6 (11)
NHR	53·2 (47)	48·4 (20)
NHL	53·1 (45)	48·5 (20)
NH'	50·9 (60)	46·5 (36)
NB	27·3 (61)	25·2 (35)
SC	7·8 (44)	8·1 (22)
SS	3·1 (44)	2·5 (21)
DC	22·5 (45)	22·1 (20)
DS	9·5 (45)	8·3 (20)
DA	31·9 (40)	30·0 (17)
O_1R	43·3 (41)	41·6 (21)
$O_1'R$	39·9 (32)	38·5 (17)
Lacr. O_1R	39·3 (38)	37·9 (18)
O_2R	34·4 (42)	33·7 (22)
O_1L	43·3 (44)	41·2 (19)
$O_1'L$	39·7 (34)	37·7 (14)
Lacr. O_1L	39·2 (40)	37·0 (16)
O_2L	34·6 (44)	33·9 (19)
G_1	50·6 (32)	47·2 (14)
G_1'	46·6 (34)	43·8 (13)
G_2	39·7 (21)	36·9 (9)
EH	11·8 (18)	8·9 (7)

B. *Indices and Angles.*

	Males	Females	Juv.
P ∠	89°·0 (37)	90°·0 (14)	89°·3 (6)
Pr. P ∠	85°·8 (51)	85°·5 (27)	87°·3 (16)
N ∠	66°·4 (51)	67°·7 (30)	68°·3 (15)
A ∠	76°·1 (51)	76°·2 (30)	76°·5 (15)
B ∠	37°·5 (51)	36°·1 (30)	35°·2 (15)
Oc. I.	59·5 (42)	60·5 (21)	59·2 (8)
$100\,B/L$	70·1 (60)	72·3 (37)	74·9 (20)
$100\,H'/L$	75·1 (62)	76·5 (40)	75·6 (19)
$100\,B/H'$	93·2 (56)	94·5 (37)	99·0 (19)
$100\,(B-H')/L$	−5·2 (56)	−4·3 (37)	−0·7 (19)
$100\,G'H/GB$	71·2 (53)	69·1 (23)	67·0 (17)
$100\,NB/NH'$	53·6 (58)	54·2 (35)	52·8 (18)
$100\,O_2/O_1R$	79·9 (41)	81·1 (21)	84·6 (7)
$100\,O_2/O_1L$	80·0 (42)	82·5 (19)	84·8 (7)
$100\,O_2/O_1'R$	86·6 (32)	88·1 (17)	91·2 (7)
$100\,G_2/G_1$	78·9 (20)	80·0 (9)	78·3 (5)
$100\,NB/NHR$	51·0 (44)	52·5 (19)	53·5 (7)
$100\,DS/DC$	42·1 (44)	38·4 (19)	40·9 (8)
$100\,SS/SC$	40·0 (44)	30·6 (21)	28·2 (7)
$100\,EH/G_2$	29·3 (18)	24·0 (7)	21·4 (5)
$100\,fmb/fml$	85·3 (52)	85·7 (30)	82·7 (7)

The numerical values given in Table II justified pooling. To the measurements of these series were added the values obtained from the skulls in the British Museum. The female series were also pooled, the results being given in Table III.

In order to furnish some additional evidence, the variabilities of a few characters (L, B, H', $100\,B/L$) were found for the whole male series. They are presented in Table IV, together with the corresponding values of three other series. In none of the three absolute measurements is there a significant difference between the "E" series and the Easter Islanders, although it must be admitted that there is strong evidence for a significant difference in the case of the cephalic index. The figures are: $\Delta_\sigma = \cdot7 \pm \cdot218$; $\Delta_V = 1\cdot26 \pm \cdot306$. Δ_V is clearly significant in the accepted sense of the term; a short calculation shows that a difference for σ of the same sign and as great as, or greater than, the one observed would only be expected in one out of 67 trials.

TABLE IV.

Variability of Easter Island Crania, Compared with Some Other Races (Males).

	Easter Islanders	Egyptians: "E" Series*	English: Farringdon St‡	Eskimos§
$L\,\sigma$	6·10 ± ·34 mm.	5·72 ± ·09 mm.	6·46 ± ·26 mm.	5·81 ± ·20 mm.
L V	3·20 ± ·18	3·09 ± ·05	3·42 ± ·12	3·08 ± ·11
$B\,\sigma$	4·74 ± ·29 mm.	4·76 ± ·08 mm.	5·90 ± ·24 mm.	4·52 ± ·16 mm.
B V	3·56 ± ·22	3·43 ± ·05	4·14 ± ·17	3·36 ± ·12
$H'\,\sigma$	4·61 ± ·28 mm.	5·03 ± ·08 mm.†	5·06 ± ·22 mm.	4·79 ± ·17 mm.
H' V	3·23 ± ·20	3·75 ± ·06†	3·90 ± ·17	3·43 ± ·12
$100\,B/L\,\sigma$	3·38 ± ·21	2·68 ± ·06	3·48 ± ·14	3·00 ± ·10
$100\,B/L$ V	4·83 ± ·30	3·57 ± ·06	4·64 ± ·19	4·21 ± ·15

The type thus established is peculiar in many ways, and it may be well to discuss some of its more salient features. Many of the peculiarities are brought out when computing the C.R.L. The values of the function α for some of those C.R.L.'s to be given later will therefore here be alluded to.

In other instances, however, the interracial variabilities will perhaps be of greater value for elucidating the place of the Easter Islanders among the races of mankind. We are still widely ignorant of such variabilities. In a few cases it has here been attempted to estimate them on an admittedly slender basis, for our knowledge of the human races is still far from being complete. A large number of series are available for Europe and Egypt (see Morant, *Biometrika*, Vols. XVII., XX[B].), and there are a few for Asia, Malaysia and Oceania (Morant, *Biometrika*, Vol. XVI.; v. Bonin, *Biometrika*, Vol. XXIII. pp. 52—113, and others), but in proportion to their area Africa and America are less well known from a craniological point of view. In order to avoid giving Europe an undue weight, only one representative of each of the groups

* Pearson and Davin: *Biometrika*, Vol. XVI. 1924, p. 338.

† For H instead of H'.

‡ Beatrix G. E. Hooke: *Biometrika*, Vol. XVIII. 1926, p. 40.

§ G. M. Morant, *Annals of Eugenics*, Vol. I. 1925—26, p. 263.

established by Morant has been included, and the attempt has been made to apply the same principle to the rest of the world. But it is readily admitted that the choice made—which will become evident from the subsequent tables—has been somewhat haphazard. The interracial variabilities may finally be found to be quite different from the values given here!

The brain-box of the Easter Islanders is fairly large, and the absolute value for the capacity does not differ much from that found for the Maori and Moriori, nor from the value given for the Ainos, but it is still significantly smaller than that for the Anglo-Saxons or the British Neolithic type ($\alpha(C) = 8{\cdot}49$, and $5{\cdot}76$, resp.). All the more astonishing is the shape of the brain-box. It is long, but not extremely so. Thus the British Neolithic type has a significantly greater length ($\alpha(L) = 8{\cdot}94$).

TABLE V.

Male Compound Indices and Length of Base for Some Races.

(Cf. Figures 1 and 2.)

Race	$100(B-H')/L$	LB
Easter Islanders	−5·2	110·5
Loyalty Islanders* ...	−5·2	104·2
New Caledonians* ...	−3·9	103·5
Veddahs	−2·7	98·1
Badari Egyptians	−1·3	99·3
Australians	−1·0	102·1
Naqada Egyptians ...	−0·6	101·4
Nepalese	−0·2	98·0
Northern Chinese... ...	0·6	99·0
Tagals	0·9	100·7
Ainos	0·9	105·4
Maori	1·3	103·9
British Neolithic	1·8	103·3
Fuegians	2·1	106·0
Dayaks	2·2	99·4
Gaboon Negroes	2·6	100·3
Moriori	2·9	105·6
Tasmanians	2·9	98·8
Tibetan B	3·5	99·2
Andamanese	3·9	92·6
Java B.-B.	4·0	97·2
Greeks	4·0	101·0
Burmese A...	4·4	98·5
Tibetan A	4·5	95·7
Aëtas	4·6	98·3
Farringdon Street (English)	6·7	100·1
Basques	6·8	99·6
English Bronze Age ...	8·1	102·9
Telengetes	12·3	99·8
Interracial Mean	+2·1	100·8
„ S.D.	3·8	3·5

* After Sarasin, F.: *Anthropologie der Neu Caledonier und Loyalty Insulaner*, Wiesbaden, 1916—22. All other values have been given in the present and previous issues of *Biometrika*.

On the other hand, the Easter Island skull is very narrow, the value for $\alpha(B)$ being significant for all the races used in this paper. This, however, is compensated for, if such an expression be permitted*, by an unusual height of the skull. It appears to be the highest known. Another striking peculiarity is the extreme length of the base of the skull; the mean for LB is again higher than that for any other race thus far measured.

It is not proposed to enter into a discussion of the cephalic index or the indices 100 H'/L and 100 B/H'; that should be postponed until a really representative

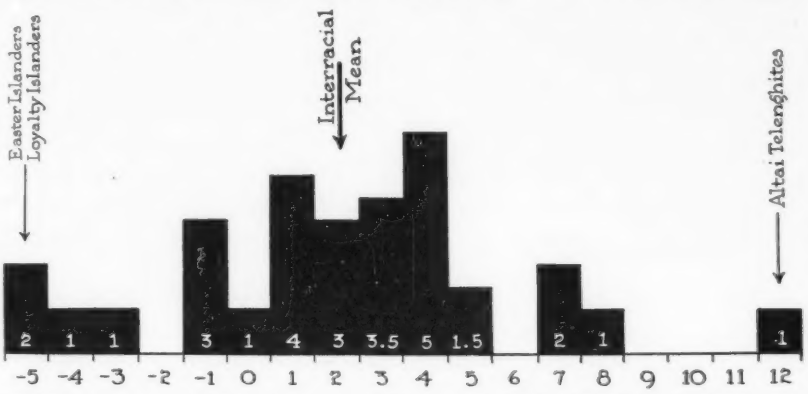


FIG. 1. Interracial Distribution of Index 100 (B - H')/L (σ = 3.8 ± .3)

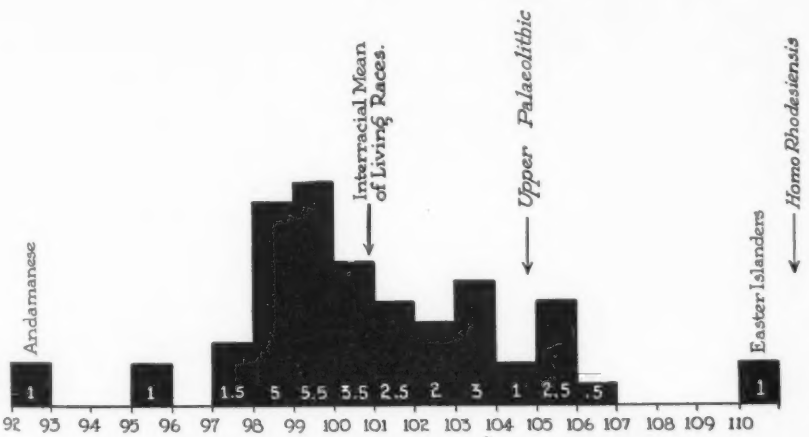

Andamanese
Interracial Mean of Living Races.
Upper Palaeolithic
Easter Islanders
Homo Rhodesiensis
1 1 1.5 5 5.5 3.5 2.5 2 3 1 2.5 .5 1
92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110

FIG. 2. Interracial Distribution of L B. (♂ means)

* This phrasing would imply that not the linear dimensions of the skulls but the capacity was directly selected.

sample of the whole of the human races is available. It may not be amiss, however, to make a few remarks about the index $100(B - H')/L$ and about the measurement LB. Their values for a number of races will be found in Table V. The provisional interracial means and standard deviations of $100(B - H')/L$ and LB are given at the foot of the table, an inspection of the entries showing that the Easter Islanders stand for these two characters right at one end of the range of distribution. Incidentally it may be remarked that the extension of the range for LB thus given somewhat alters the place for the Rhodesian skull which now becomes less far removed from modern man. The accompanying histogram (Fig. 2) is intended to illustrate this point.

It is noteworthy that the occipital index differs but little from that of races which are generally spoken of as being "higher," the difference in that character between the Anglo-Saxons for example and the Easter Islanders being insignificant (α (*Oc. I.*) = 2·96). Turning next to the face, there is also very little difference between various races in the measurements $G'H$ and J. Profound differences on the other hand are to be observed in the shape of the nasal aperture. The nose of the Easter Islanders is absolutely as well as relatively broad, and the nasal index consequently high.

Whereas the orbits show no outstanding feature, the palate is somewhat short, although not shorter for instance than those of the Anglo-Saxons or the Kham Tibetans.

The profile angle is large. Here, the Easter Islanders are more in line with the Anglo-Saxons than with fringe races.

Following up the work of Lüthy, the study of the profile may be extended by taking the profile angle of the nasal roof into consideration ($NRP\angle$ of Table VI). Lüthy provided means for a number of races, most of which were based on more than 20 skulls. His angle of the nasal roof is the acute angle which the line NL (in the usual notation of the Biometric School) forms with the Frankfurt horizontal, and which can easily be found from the tables usually given in *Biometrika* for the median sagittal contours as the complement to the angle γNL. Unfortunately Lüthy has given the *Pr.* $P\angle$ and not the $P\angle$ to the Alveolar Point.

For an attempt to find a correction to be applied to the $P\angle$ for converting it into the *Pr.* $P\angle$, the following values are available:

	$P\angle$	*Pr.* $P\angle$	Difference
Dayaks ...	84°·7	82°·7	2°·0
Java B.-B. ...	84°·9	81°·1	3°·8
Tagals	87°·0	84°·2	2°·8
Aëtas	87°·5	85°·0	2°·5
Andamanese ...	85°·0	81°·8	3°·2

Taking the mean of these and applying it as the sought correction seemed unreliable. A rough calculation led to a mean difference of $2°·9 \pm 0°·26$; and there was nothing

else to be done but to leave out a number of series! All that could safely be gathered together are given in Table VI, largely based on Lüthy's unsexed series. Working from the interracial means, the coefficient of correlation between the two angles has been worked out. It is quite insignificant, but the interracial standard deviations are interesting. It appears from these values given at the foot of Table VI that

TABLE VI.

Profile Angle of Face and of Nasal Roof for some Races.

Measured in Degrees.

	1. *Pr. P* ∠	2. *NRP* ∠	1 – 2
Australians	76·8	61·6	15·2
Papuans	77·2	60·1	17·1
Cameroon	78·1	64·1	14·0
N.E. Africans ...	78·3	64·1	14·2
Java B.-B.* ♂ ...	81·1	66·9	14·2
Tamils	81·7	57·9	23·8
Burmese	82·0	66·8	15·2
Battak	82·1	67·9	14·2
Singhalese	82·1	59·0	23·1
Northern Chinese† ♂	83·4	64·4	19·0
Egyptians	83·7	55·6	28·1
Tagals* ♂	84·2	71·7	12·5
Veddahs	84·3	62·0	22·3
Aëtas* ♂	85·0	68·5	16·5
Farringdon Street‡ ♂	85·8	55·2	30·6
Easter Islanders ♂ ...	85·8	72·7	13·1
Bündner	87·0	53·2	33·8
Interracial Mean ...	82·27	63·04	19·23
„ S.D. ...	3·03	5·55	6·40

Most of these values are taken from Lüthy, *loc. cit.*, and are from unsexed material.

the profile angle of the nasal roof is a useful character for racial comparisons. This confirms Lüthy's opinion though he based his judgment on the intraracial standard deviations. But as these were worked out from a mixed material of both male and female skulls, they appear hardly sufficiently reliable. Our own material, consisting as it did of only 12 skulls, is entirely insufficient for this purpose.

It seemed of interest to compare the Easter Islanders with other races in regard to the proportion of the facial area to the brain-box. This was done in the way shown by Morant, by forming an index from the area of the fundamental triangle of the face and the square of the sagittal arc *S*—in other words by computing the

* After von Bonin: "Kraniologie von Ostasien," *Biometrika*. Vol. XXIII. pp. 52—113.

† After Davidson Black: "A Study of the Kansu and Honan...Skull, etc.," *Palaeontologica Sinica*, Series D, Vol. VI. fasc. i, 1928.

‡ After Hooke, *Biometrika*, Vol. XVIII., taken from type contours.

value of the expression $100\,A/S^2$. The results are given in Table VII. It is not yet clear how much can be deduced from these values. Obviously, this index fails to arrange the races in anything approaching an intelligible order. On the other hand, the interracial variability is not low (a rough computation by slide-rule led to an interracial standard deviation of about ·13, which gives a v of between 5 and 6), and it might be well worth while to follow up this matter further.

TABLE VII.

Area of Facial Triangle and Length of Sagittal Arc (Male Crania).

	A	S	$100\,A/S^2$
Veddahs	2788	363·1	2·11
English Bronze Age ...	3215	376·6	2·12
Andamanese	2658	349·0	2·18
British Neolithic	3299	389·0	2·18
Upper Palaeolithic ...	3450	398·3	2·18
Basques	3098	375·2	2·20
Badari Egyptians	3053	372·0	2·21
Farringdon Street (English)	3185	378·8	2·22
Congo Negroes	2917	361·8	2·23
Naqada Egyptians ...	3224	372·6	2·32
Loyalty Islanders	3492	386·9	2·33
Australians	3236	371·0	2·35
Aëtas	3126	360·0	2·41
Greeks	3258	367·2	2·42
Easter Islanders	3477	378·9	2·42
New Caledonians	3432	374·4	2·45
Telengetes	3302	357·1	2·59
Mean			2·29

Values taken from previously published tables in *Biometrika* and *Annals of Eugenics*, except for New Caledonians and Loyalty Islanders, which were taken from Sarasin, *loc. cit.*, Table IV.

In several instances the Easter Islanders were found to be well near one end of the range of interracial distribution for a given character. Such a type may justly be called a specialized one. It can then be asserted that the Easter Islanders are a specialized—but by no means a primitive—type in regard to two things at least, the configuration of their brain-box [$100\,(B-H')/L$ and LB] and their facial profile, as revealed by the profile angle of the nasal roof. How far they deviate in these respects from other races in the South Sea cannot yet be said. It would appear that the inhabitants of the Fiji Islands which have been studied and observations concerning which have been published by Flower are somewhat similar to them. Howsoever that may be, the specialization observed in a given insular type would well fit in with a theory of evolution recently advanced by Pearson (1930). A small group may be supposed to have landed on the island; inbreeding has by

necessity gone on since then and the result should be an aberrant type such as we meet with, provided the original settlers did not happen to represent the exact racial mean of the stock they hailed from!

It is interesting to compare the shape of the male skulls with that of the female and juvenile skulls, figures for which are also included in Table III. It becomes clear then that in regard to the shape of the brain-box at least the males are further removed from the interracial mean and the juveniles are closer to it than the females. It may also be noteworthy that the $N\angle$ becomes smaller as one goes from juveniles to females and males. The values given for various races by Morant (1927, p. 335) should be consulted. It is seen then that most South Sea Islanders have a very large $N\angle$, and it would appear that the comparatively small value for the Easter Islanders might be considered as a secondary change.

Easter Island being situated far removed from the centre of dispersion of the human race, the question arises whether its inhabitants show any relation to other fringe races who are generally considered as remnants of formerly widespread populations, pushed into "corners" by subsequent waves of emigration arising near the centre of dispersal. C.R.L.'s between a number of primitive races and the Easter Islanders have been computed and are given in Table VIII. The values for the

TABLE VIII.

C.R.L.'s between Easter Islanders (49·7) *and Fringe Races* (*Males*).

Race	All Characters	No. of Characters	Indices and Angles	No. of Characters	Reduced value for all Characters
Moriori (33·9) ...	26·16 ± ·17	30	35·98 ± ·28	12	64·90 ± ·42
Maori (43·0) ...	21·46 ± ·19	25	36·44 ± ·36	7	46·54 ± ·41
Tibetan B (14·5) ...	15·01 ± ·17	31	17·04 ± ·28	12	66·85 ± ·76
Eskimos (186·7) ...	50·14 ± ·19	26	65·66 ± ·30	10	63·88 ± ·24
Ainos (79·6) ...	37·74 ± ·19	25	63·06 ± ·32	9	61·67 ± ·31

C.R.L.'s between Easter Islanders and European Races.

Upper Palaeolithic (10·9)	17·43 + ·23	16	11·10 ± ·38	5	97·49 ± 1·3
British Neolithic (34·5)	27·79 ± ·19	24	28·73 ± ·32	8	68·22 ± ·47
Anglo-Saxons (35·3)	25·94 ± ·17	31	36·63 ± ·26	12	62·83 ± ·41

Maori, Moriori, and Ainos are taken from Morant (1922—23), the values for the Eskimos have been given by the same author in his paper on the Chancelade skull (1925—26). In a few instances the number of individuals on which the averages were based were not to be found in this paper, they were ascertained from the original memoir by Fürst and Hansen.

The Easter Islanders are almost equally remote from all the fringe races here considered. It should be mentioned, however, that in the case of the Eskimos the

nasal characters *NB* and 100 *NB*/*NH'* contribute almost half of the total value of the C.R.L. Even allowing for this, the value would remain too high to form a basis for the assertion of any close relationship.

Next to the fringe races, the C.R.L. has been computed for the Upper Palaeolithic race, the means of which have become known through Morant's work (1930) and for the British Neolithic and the Anglo-Saxons (1926). These races have been chosen here because the opinion has been frequently expressed (by de Quatrefages and others) that there is a racial affinity between the "Caucasian" race, to which Morant's group "*D*" (1928) perhaps conforms best, and that inhabiting the South Sea. There is no justification for this view as far as the Easter Islanders and the two races of group "*D*" here used are concerned.

TABLE IX.

Mean Measurements of 12 *Male Easter Islanders, Transverse Contours.*

	MA	1	2	3	4	5	6
R	121·2	55·8	57·8	60·2	62·8	63·4	62·8
L		61·2	63·4	65·2	67·4	68·1	67·2

	7	8	9	10	$A\frac{1}{4}$	*ZRy*	*ZRx*
R	61·2	57·2	48·7	33·9	17·3	58·4	3·8
L	65·9	62·4	54·2	39·1	21·7	64·4	4·4

TABLE X.

Mean Measurements of 12 *Male Easter Islanders, Horizontal Contours.*

	FO	$F\frac{1}{4}$	$F\frac{1}{2}$	2	$2\frac{1}{2}$	3	4	5
R	187·2	23·6	36·4	47·8	46·3	48·2	54·0	60·1
L		24·3	35·3	47·4	46·4	48·1	53·3	61·0

	6	7	8	9	10	$O\frac{1}{4}$	*Ty*	*Tx*
R	64·3	65·3	62·4	55·5	43·2	25·2	49·0	20·0
L	65·9	66·5	62·9	55·7	42·8	24·8	49·0	20·4

A hint might be added as regards the type contours. As has already been stated the type contours of the Easter Islanders have been derived solely from the Leiden skulls, and are therefore not strictly representative of the whole series. Even so a comparison with the Anglo-Saxons, the contours of which have been given by Morant (1926), is interesting in several ways. Superposing them so that the Frankfurt horizontal and the point *N* coincide, the brain-boxes of the two races look very dissimilar, whereas the Alveolar Points almost cover each over. If, however, the

contours be turned in such a way as to bring N and β respectively together, much of the difference in the pattern of the brain-box disappears. It should be noted in the first place that now the glabella-inion line—which can be made out even in the Easter Islanders contours—shows practically no difference in its location in the two races, circles indicating the positions of its end points would overlap. Furthermore, the outlines of the brain-boxes cover each other for practically the whole part between the nasion and the vertex. On the other hand, the position of the Alveolar Point now shows marked differences, emphasizing the peculiar "drawn in" face of the Easter Islanders. Two points seem to emerge from these somewhat casual observations: there appears to be a greater variety in the moulding of the occipital region

TABLE XI.

Mean Measurements of 12 *Male Easter Islanders, Median Sagittal Contours.*

$N\gamma$		181·2	Lambda:	x from γ	10·5
Ordinates above $N\gamma$:	0=N	39·3		y	45·4
	$N\frac{1}{4}$	50·4	Suborbital point:	x from N	9·2
	1	67·7		y	24·9
	2	81·0	Auricular point:	x from N	92·6
	3	88·9		y	25·1
	4	93·0	Opisthion:	x from N	131·8
	5	95·8		y	46·1
	6	96·3	Basion:	from γ	96·7
	7	91·3		from N	107·2
	8	80·6	Alveolar point:	from *Bas.*	98·8
	9	60·4		from N	67·2
	$\gamma\frac{1}{4}$	34·4	Sp.:	x from N	71·0
	$\gamma\frac{2}{4}$	28·4		y	34·9
Ordinates below $N\gamma$:	1	54·2	N. S.:	x from N	2·1
	2	52·3		y	50·5
	P'	52·4	Prosthion:	x from N	3·4
	8	39·9		y	63·8
	9	27·5	Frontal Arc:	x from N	50·2
	$\gamma\frac{1}{2}$	20·4		y	27·2
	$\gamma\frac{1}{4}$	14·1	Occipital Arc:	x from λ	49·6
Vertex:	x from N	97·8		y	28·8
	y	97·1	Palate P:	x from Alv. P	34·9
Bregma:	x from N	67·4		y	14·6
	y	91·9	Nose:	$\angle LN\gamma$	107°·3
Glabella:	x from N	5·3		NL'	16·0
	y	12·9		NL	16·7
Occipital point:	x from γ	1·3			
	y	9·6			

than in that of the frontal part of the skull, thus vindicating once more the usefulness of the occipital index of Pearson. In the second place, the writer ventures to think that a renewed study of the orientation of the skull might reward the labour involved.

Turning next to the horizontal contours, the frontal part is very much alike in both races. The general pattern of the outline is also identical in both cases, the Easter Islanders show, just as all other modern races, the greatest breadth at the seventh ordinate. The fact that the Easter Islanders show an asymmetry of the skull reverse to that seen in the Anglo-Saxons may be noticed in passing.

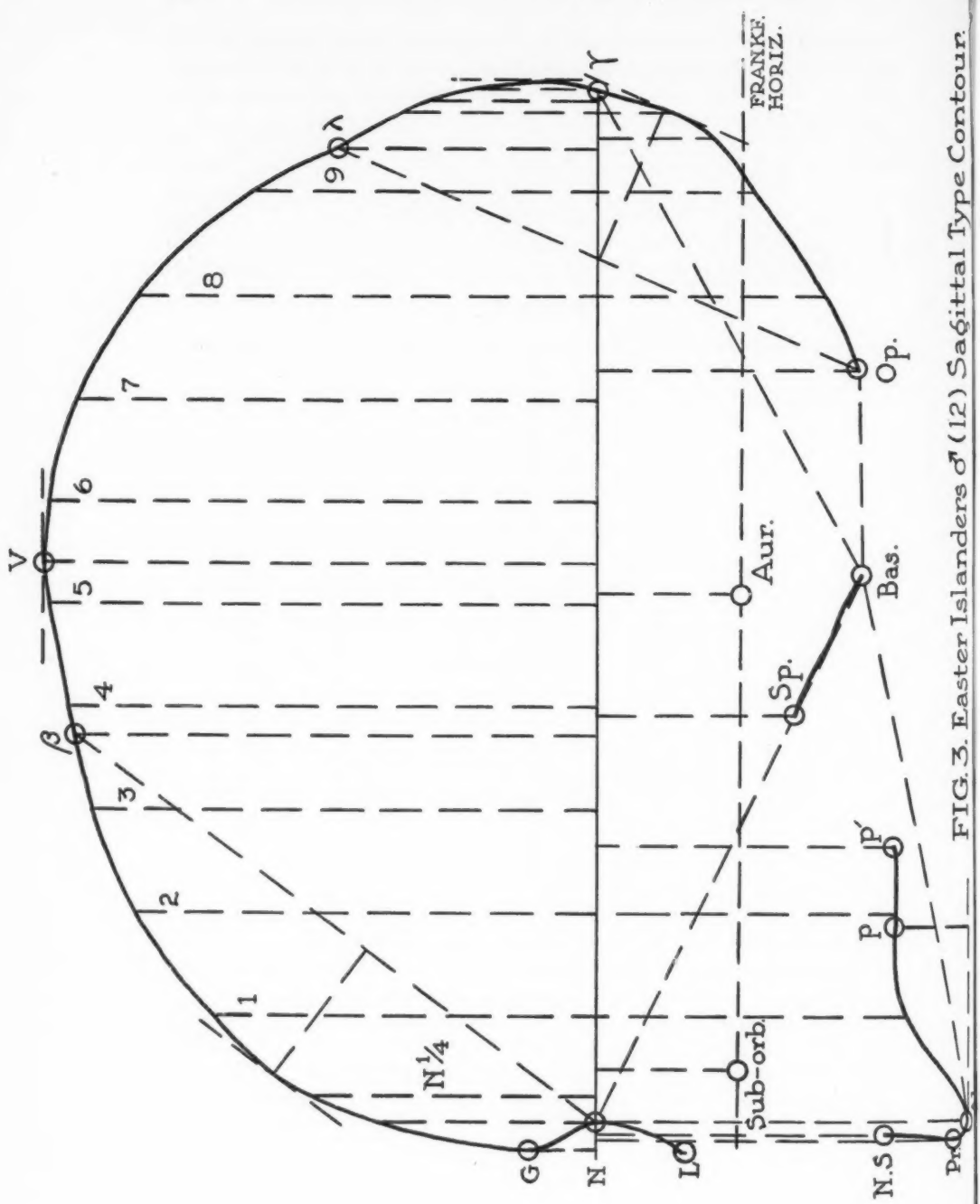
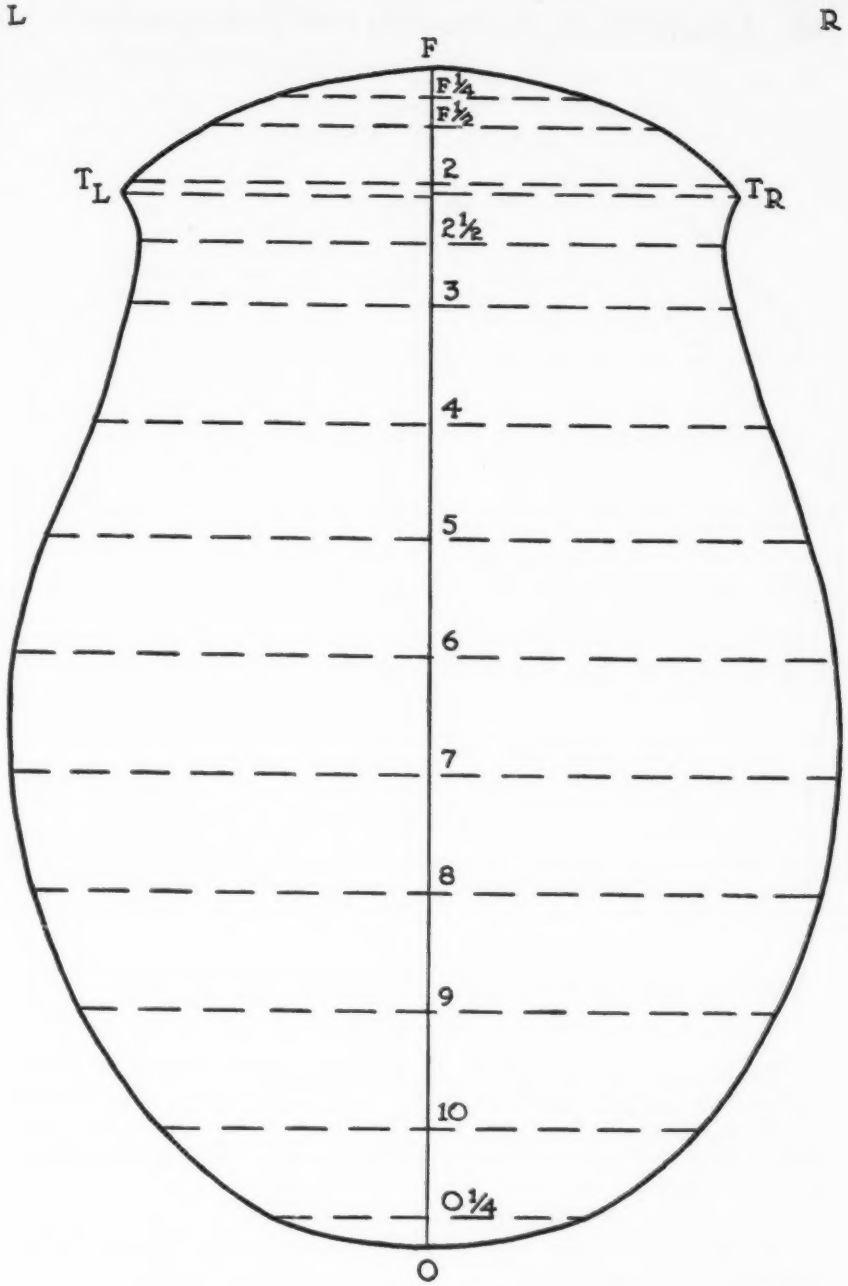
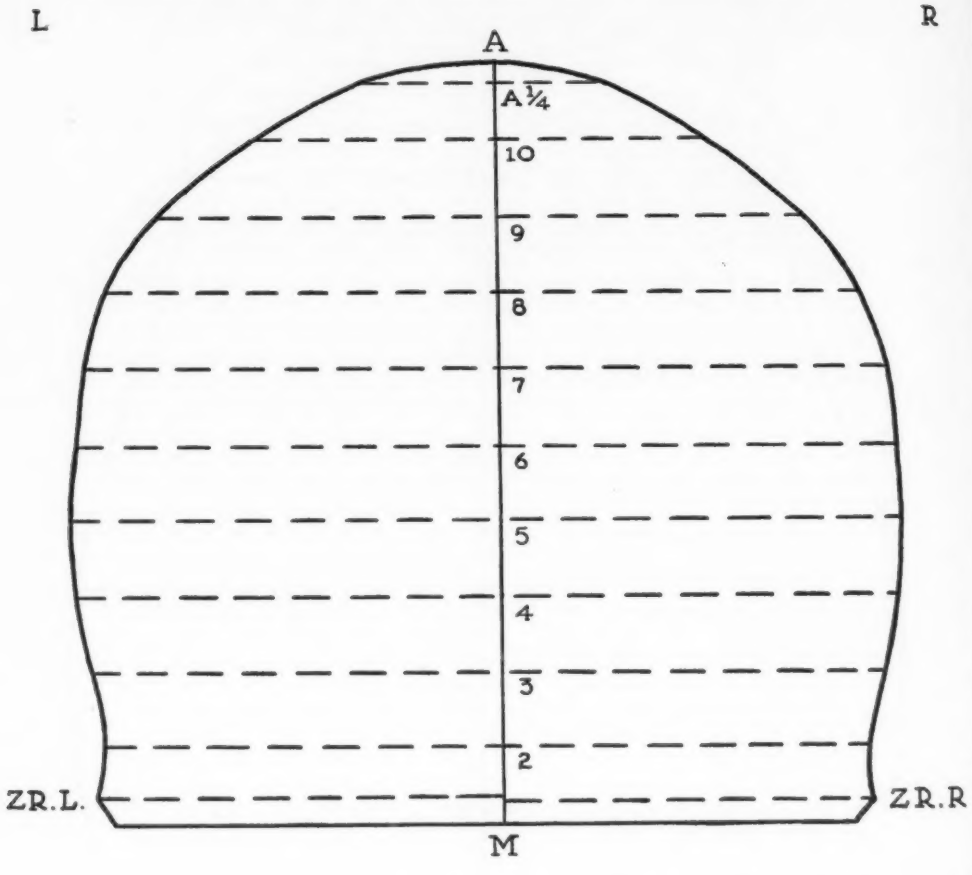


FIG. 3. Easter Islanders ♂ (12) Sagittal Type Contour.



Easter Islanders ♂ (12) Horizontal Type Contour.
FIG. 4.

FIG. 3. Easter Islanders ♂ (12) Sagittal Type Contour.



Easter Islanders ♂ (12) Transverse Type Contour.
FIG. 5.

This character appears again in the transverse contour. Here it should be noticed in the first place that in spite of the great narrowness of the skull, the outline is again conforming to the pattern usually found in modern races: the greatest width is at about the fifth ordinate, and the narrowing near the zygomatic ridges is well pronounced.

To sum up the following conclusions may be stated:

1. There is no statistical reason to doubt the homogeneity of the craniological material coming from Easter Island.

2. In several respects the type defined in this paper appears to be far removed from the interracial mean, thus justifying its description as a specialized race.

3. There is no racial affinity to any of the fringe races which have been described thus far.

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P.S. While crossing the Atlantic on my way back to America I came across a book in the ship's library: Schulze-Maizier: *Die Osterinsel* (Insel-Verlag, Leipzig, no date). Evidently intended for a wide public, it is popular science at its best. It appears from this book that Weisser was paymaster under Captain-Lieutenant Geiseler on board the "Hyäne," and not a dealer as the letter from Professor Juinboll cited in the footnote, p. 249, suggests. It follows that the skulls now in Leiden must have been collected at the same time as those reported on by Volz.

Note to Table VIII.

By G. M. MORANT, D.Sc.

It it stated in the opening paragraph of this paper that the writer intends to postpone a survey of the craniology of the South Sea Islands until more material will be available, but, until this is forthcoming, it appears to be hazardous to stress the suggestion that the type of the Easter Islanders is a peculiarly specialised one. The lowest reduced coefficient of racial likeness given is 46·54 ± ·41 with the Maori. There appears to be no other cranial series available, except the Andamanese, having its lowest reduced coefficient greater than 23, and there is generally no difficulty in finding one or more values less than 10. It seemed worth while to make comparisons between the Easter Islanders and some other series not dealt with in Table VIII. Some values found for male means are given in the table below.

The Loyalty Islanders appear to resemble the Easter Islanders far more closely than the latter do any of the types with which they are compared in Table VIII. The only marked differences between the two types are for the profile angle—the Loyalty Islanders being prognathous—and *LB*. If material were available for other islands of the Pacific nearer to Easter Island, it is extremely probable that still closer connections would be found.

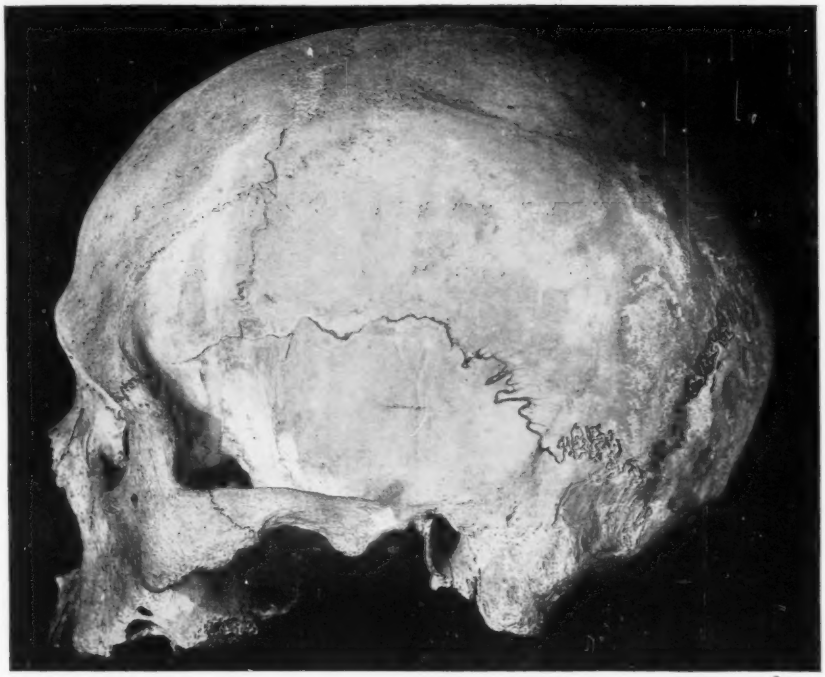
	Crude Coefficients with Easter Islanders (50·3—54·4)				Reduced coefficients
	All Characters	No. of Characters	Indices and Angles	No. of Characters	
Loyalty Islanders* ... (33·8)	9·19 ± ·22	18	14·40 ± ·39	6	22·04 ± ·54
Patagonians: Ancient† (48·7)	15·62 ± ·19	25	17·32 ± ·36	7	31·57 ± ·39
New Caledonians* ... (89·5)	28·85 ± ·22	18	47·62 ± ·39	6	42·64 ± ·33
Fuegians‡ ... ... (33·7)	30·42 ± ·19	26	40·50 ± ·30	10	73·23 ± ·45
Patagonians: Modern† (35·6)	33·64 ± ·19	25	46·24 ± ·36	7	80·68 ± ·46

It is interesting to note that the coefficient between the male Fuegian (36·7) and Modern Patagonian (35·5) series is as low as 4·01 ± ·21 for 20 characters, and this leads to a reduced value of 11·14 ± ·59. The only α's greater than 10 are for J (24·40), 100 H'/L (12·33) and C (10·21), while it is the Patagonian type which has a bizygomatic breadth (148·0) which is apparently the largest male mean yet recorded.

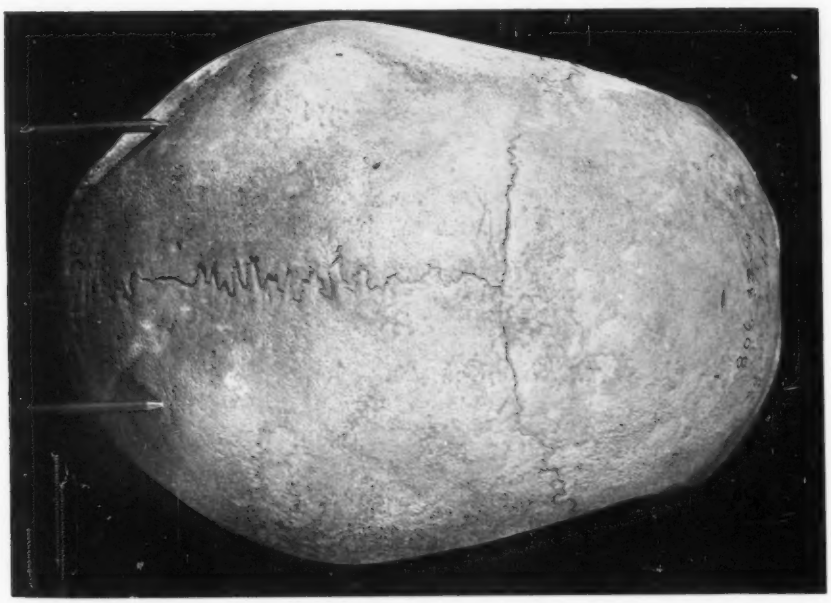
* Sarasin, F., *Anthropologie der Neu Caledonier und Loyalty Insulaner*, Wiesbaden, 1916—22.

† Marelli, "Contribución a la Craneología de la Primitivas Publaciones de la Patagonia (Observaciones morfobiométricas)," *Anales del Museo Nacional de Historia Natural de Buenos Aires*, t. xxvi. pp. 31—91, 1913. All deformed skulls were excluded in re-computing the means.

‡ Unpublished pooled means.

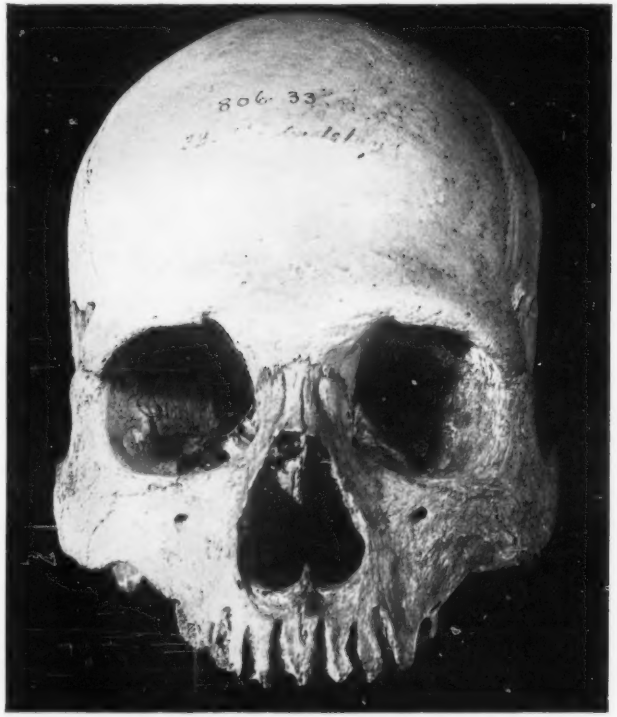


A. *Norma lateralis* (left profile) (*circa* 0·6 natural size).



B. *Norma verticalis* (*circa* 0·6 natural size).

Normal Male Easter Island Skull (R. C. S. No. 806. 33)



C. *Norma facialis* (*circa* 0·7 natural size).

D. *Norma occipitalis* (*circa* 0·65 natural size).

Normal Male Easter Island Skull (R. C. S. No. 806. 33)

		C	*L*	*F*	*B*	*H′*	*H*	*OH*	*Bregmatic OH*	*LB*	*B′*
R.C.S.	806·00	1340	188	188	130	137	136	118·5	117	106	91·6
	·01	1355	183	183·5	128	142·5	143	125	124	108	86·2
	·02	—	204·5	204	—	—	—	—	—	—	95·0
	·06	—	189	186·5	—	—	—	—	—	—	—
	·08	1530	194	189	130·5	—	145	125	126·5	112·5	95·5
	·10	1365	179·5	178	141	140	141	125·5	124·5	113	99·0
	·11	1470	184	181	139·5	138·5	140	124·5	123	101	100·2
	·15	1525	190	189	128	148·5	151	130	127	108	95·5
	·16	1680	195·5	192	140	147	147	129·5	129·5	117	93·0
	·17	—	191	186	—	—	—	120	121·5	—	93·2
	·19	1375	201	200	131·5	146	146·5	126	126	114	99·3
	·21	—	191·5	190	—	—	—	—	—	—	97·0
	·26	—	193·5	191	132·5	144	144	125·5	125·5	113·5	91·0
	·27	—	187	186	—	136	136·5	116·5	117	108	87·0
	·28	1560	189	186	133	145	149	127	123	110	96·0
	·29	—	197	193	136·5	144·5	145	126	125	110	97·8
	·30	—	189	188·5	125	—	—	—	—	—	88·0
	·31	1555	193	190	139	144	147·5	125·5	123	112	91·6
	·32	1240	182·5	181	129	137·5	138·5	117	116	106·5	90·0
	·33	1435	191	188	133·5	143	144	126	125·5	109·5	95·0
	·34	—	187	185·5	—	—	—	—	—	—	94·8
	·35	—	196	194	127·5	139	140·5	118	117	118	94·0
	·36	—	193	193	—	138	139	121	120·5	106	—
	·37	1460	192·5	191·5	133·5	145	146·5	124	124	108	94·0
	·38	—	189	188	—	142	142	120	120	109	—
	·39	—	200	196	—	145	147	127	126	116	92·0
	·40	1600	206	200	133·5	148	149·5	127·5	126	116	89·3
	·41	—	193·5	194	—	—	—	119	120	—	93·0
	·45	1485	190	188·5	136	145	148	126·5	124	110	90·7
	·46	—	200	199·5	131	147	148·5	132·5	132	111	91·0
	·47	1425	190	188·5	125	143	149	128·5	125	112	92·8
	·50	—	192	189	134·5	146·5	150	126·5	125	112	94·0
	·51	1525	199	198	128	144·5	146·5	119	120·5	111	93·0
	·52	—	201·5	198	136·5	153·5	155·5	129	127·5	117	—
	·53	—	184	182	—	141	141	120·5	120·5	115	94·6
	·56	1585	197	195	133	144·5	145·5	123·5	122·5	115	105·0
Leiden	192	—	189	188·5	133	140·5	141	121	121	107·5	94·2
	196	—	197	196	133·5	147	147·5	128	126·5	117·5	—
	198	—	181	181	132·5	138·5	139·5	120·5	120	104·5	92·2
	199	—	188	188	136	—	—	119	121	—	95·0
	201	—	198	196·5	140	145	146	128	126·5	112·0	90·8
	202	—	193	194	130·5	140·5	139	124·5	124	103·7	94·6
	203	—	191	193	134·5	136·5	141	121	118·5	—	98·5
	204	—	181·5	180	126	139	145	126	122	105·0	86·6
	205	—	191	190	134·5	146	150	120·5	117·5	118·5	91·5
	207	—	183·5	183	126	135·5	134	112	113	106·2	89·0
	208	—	190·5	186·5	134·5	141	142·5	123	120·5	111·8	92·1
	209	—	188	186	142	154	156	131·5	130	111·1	98·9
	210	—	191	190	131·5	134·5	133·5	116	117·5	102·0	91·2
B.M.	1	—	193	190·5	128	153	—	—	—	111·8	90·4
	3	—	184	184	124·5	137·5	—	—	—	108·0	90·5
	6	—	186	186·5	124	138·5	—	—	—	108·2	92·2
	9	—	196·5	192·5	130·5	—	—	—	—	—	92·0
	10	—	188	189	—	133·5	—	—	—	106·5	93·0

LB	*B′*	*U*	*Glabella U*	*Q′*	*Bregmatic Q′*	S_1'	S_2'	S_3'	S_1	S_2	S_3	*S*	*fml*	*fmb*	*G′H*	*J*	*GB*
106	91·6	511	519	313	306	108·2	121·4	96·0	126	136	114·5	376·5	35·0	27·0	65·6	131	98·2
108	86·2	501	506	323	315	114·0	105·4	99·6	133·5	115	117·5	366	36·5	29·4	68·0	124	88·0
—	95·0	—	—	—	—	122·2	124·2	—	146	138	—	—	—	—	73·3	—	97·9
—	—	—	—	—	—	115·0	128·2	—	131·5	143	—	—	—	—	—	—	—
112·5	95·5	517	530	321	321	116·4	116·8	99·0	134·5	129	118	381·5	39·0	32·9	73·4	143	105·0
113	99·0	508	517	336	329	113·6	102·6	103·0	129	115	120	364	29·5	24·5	—	136·5	102·3
101	100·2	514	529	329	326	122·5	111·0	93·6	139	124·5	113·5	379	34·0	28·8	76·0	135	100·0
108	95·5	514	519	329	322	114·3	121·0	100·0	130	137	119	386	—	32·0	66·1	135	94·3
117	93·0	535	544	346	341	126·0	110·0	97·5	146	121·5	117	384·5	38·3	32·4	73·3	136	97·0
—	93·2	—	—	—	311	120·6	115·0	—	137·5	128	—	—	—	—	—	—	—
114	99·3	534	545	—	—	111·0	122·0	100·0	129·5	135	127	391·5	—	30·4	74·9	137	101·8
—	97·0	—	—	—	—	117·5	122·7	—	137	134	—	—	—	—	—	—	—
113·5	91·0	520	536	326?	322?	113·8	109·2	—	129	119	—	—	—	29·6	72·7	—	94·2
108	87·0	503?	511?	—	—	111·0	108·4	101·2	129	—	122	—	—	—	64·2	—	94·2
110	96·0	512	523	325	316	108·5	124·7	94·3	120·5	144	113	378	37·4	31·3	—	132	95·1
110	97·8	536	546	332	328	118·0	—	—	138	—	—	395?	—	—	74·0	140·5	102·0
—	88·0	—	—	—	318	115·0	111·0	106·5	132·5	123	130	385	—	—	—	—	—
112	91·6	525	534	335	329	111·6	115·2	100·8	125	131	124	381	39·0	30·0	74·0	—	90·0
106·5	90·0	495	505	310	303	109·5	110·0	95·0	125	120	115	360	35·0	28·2	65·8	133	100·0
109·5	95·0	517	526	330	325	120·0	111·7	104·3	139	122	126	387	36·1	30·1	73·0	129·5	99·7
—	94·8	—	—	—	—	112·4	—	—	129	—	—	—	—	—	63·0	—	—
118	94·0	514	535	310	307	115·0	101·0	103·0	132	110	134	377	—	—	71·8	—	105·4
106	—	—	—	—	—	114·5	119·0	95·0	136	133	116	385	38·0	30·1	67·0	—	101·7
108	94·0	530	535	330	326	119·4	117·5	93·8	140	130	115	385	35·9	28·9	—	136	98·0
109	—	—	—	—	—	116·2	112·0	97·6	134	124	118	376	38·7	—	—	—	—
116	92·0	525	533	328	325	122·0	121·0	94·0	141	141	116	398	35·4	32·5	—	134	95·6
116	89·3	536	557	328	325	122·5	114·6	109·0	143	126	134	403	37·9	33·1	69·0	139·5	108·6
—	93·0	520	—	307	310	118·3	113·0	—	—	124	—	—	—	—	74·5	133	97·6
110	90·7	513	530	333	325	116·0	104·0	112·0	132	114	135	381	35·2	30·1	70·8	136·5	100·0
111	91·0	533	541	338	334	120·4	128·3	103·0	140	145	123	408	39·5	28·5	—	—	—
112	92·8	512	519	328	321	113·0	109·2	103·0	135	124	124	383	37·3	33·0	70·5	135	99·4
112	94·0	523	535	332	325	116·0	121·0	—	133	138	—	—	—	—	70·0	—	—
111	93·0	529	538	311	311	114·7	127·0	104·0	136	143	126	405	34·5	28·2	69·0	131	100·2
117	—	534	555	336	327	123·0	116·8	109·5	142	130	137	409	35·7	32·9	70·6	141	105·0
115	94·6	501	511	—	—	110·2	106·3	100·6	125	115	122	362	33·1	30·7	71·1	—	95·4
115	105·0	530	538	322	319	119·8	97·0	—	139	104	—	—	—	—	73·4	142	98·2
107·5	94·2	517	520	328	326	112·0	117·6	99·7	130	132·5	122	384·5	34·2	29·0	61·6	129	92·3
117·5	—	—	—	—	—	119·9	117·2	100·2	138	127	120	385	40·3	36·1	73·2	—	—
104·5	92·2	503	506	324	320	110·0	106·2	106·3	126	116	129	371	33·4	30·3	70·0	131	89·3
—	95·0	513	520	319	316	118·0	111·0	88·4	141	122	112	374	—	—	—	—	93·0
112·0	90·8	536	545	341	338	123·5	111·5	105·2	144	125	131	400	40·0	32·8	68·3	136	100·7
103·7	94·6	525	529	326	322	116·9	124·3	104·5	134	139	124	397	36·2	31·6	61·7	132	89·0
—	98·5	530	530	315	308	—	122·5	106·0	—	138	128·5	—	30·8	27·3	—	—	—
105·0	86·6	495	502	323	318	111·5	110·4	95·0	127	126	108	362	40·4	30·9	69·8	131	97·4
118·5	91·5	521	532	317	316	107·5	120·0	87·5	123	140	104	367	—	—	77·0	142	102·0
106·2	89·0	495	506	298	302	116·3	106·7	90·7	134	117·5	107	358·5	35·0	32·0	66·5	128	95·0
111·8	92·1	519	525	324	317	111·2	105·4	104·7	127	117	127·5	371·5	37·0	30·2	66·2	138	99·2
111·1	98·9	523	531	339	337	114·5	118·0	104·8	130	133	120	383	38·6	34·0	68·5	138·5	92·5
102·0	91·2	516	524	312	310	112·0	113·2	105·8	130	124	129	383	36·4	29·5	60·8	128	102·4
111·8	90·4	516	531	—	325	123·8	107·2	116·6	142	115	140	397	35·8	30·0	66·5	132·5	100·0
108·0	90·5	503	510	—	292	105·0	111·8	99·3	119·5	124·5	120	364	35·5	32·5	67·0	127·5	94·8
108·2	92·2	505	510	—	307?	112·3	118·1	92·3	129·5	132	110	371·5	34·0	31·0	—	—	—
—	92·0	—	—	—	—	115·2	122·5	—	132	138	—	—	—	—	—	—	—
106·5	93·0	—	—	—	—	111·0	108·0	99·2	129	120	126	375	34·0	29·9	65·0	131	95·2

TABLE XII. *Indi*

Glabella U	*Q'*	*Bregmatic Q'*	S_1'	S_2'	S_2'	S_1	S_2	S_2	*S*	*fml*	*fmb*	*G'H*	*J*	*GB*	*GL*	*PH*	*NHR*	*NH*
519	313	306	108·2	121·4	96·0	126	136	114·5	376·5	35·0	27·0	65·6	131	98·2	101·0	17·3	49·0	49·0
506	323	315	114·0	105·4	99·6	133·5	115	117·5	366	36·5	29·4	68·0	124	88·0	104·8	20·4	51·0	51·0
—	—	—	122·2	124·2	—	146	138	—	—	—	—	73·3	—	97·9	—	21·0	54·5	54·1
—	—	—	115·0	128·2	—	131·5	143	—	—	—	—	—	—	—	—	—	54·1	—
530	321	321	116·4	116·8	99·0	134·5	129	118	381·5	39·0	32·9	73·4	143	105·0	107·0	15·7	59·2	59·0
517	336	329	113·6	102·6	103·0	129	115	120	364	29·5	24·5	—	136·5	102·3	—	—	55·0	55·0
529	329	326	122·5	111·0	93·6	139	124·5	113·5	379	34·0	28·8	76·0	135	100·0	93·6	18·3	57·5	57·5
519	329	322	114·3	121·0	100·0	130	137	119	386	—	32·0	66·1	135	94·3	99·8	15·4	53·3	54·3
544	346	341	126·0	110·0	97·5	146	121·5	117	384·5	38·3	32·4	73·3	136	97·0	101·3	15·1	60·3	60·3
—	—	311	120·6	115·0	—	137·5	128	—	—	—	—	—	—	—	—	—	—	—
545	—	—	111·0	122·0	100·0	129·5	135	127	391·5	—	30·4	74·9	137	101·8	111·0	19·9	55·5	54·9
—	—	—	117·5	122·7	—	137	134	—	—	—	—	—	—	—	—	—	—	—
536	326?	322?	113·8	109·2	—	129	119	—	—	—	29·6	72·7	—	94·2	109·0	—	53·0	53·3
511?	—	—	111·0	108·4	101·2	129	—	122	—	—	—	64·2	—	94·2	100·3	14·8	50·5	50·3
523	325	316	108·5	124·7	94·3	120·5	144	113	378	37·4	31·3	—	132	95·1	—	—	52·8	51·1
546	332	328	118·0	—	—	138	—	—	395?	—	—	74·0	140·5	102·0	108·3	17·7	57·4	54·3
—	—	318	115·0	111·0	106·5	132·5	123	130	385	—	—	—	—	—	—	—	—	—
534	335	329	111·6	115·2	100·8	125	131	124	381	39·0	30·0	74·0	—	90·0	107·5	19·7	58·1	58·0
505	310	303	109·5	110·0	95·0	125	120	115	360	35·0	28·2	65·8	133	100·0	105·2	14·8	51·5	—
526	330	325	120·0	111·7	104·3	139	122	126	387	36·1	30·1	73·0	129·5	99·7	105·6	17·0	58·9	57·1
—	—	—	112·4	—	—	129	—	—	—	—	—	63·0	—	—	—	—	49·0	49·1
535	310	307	115·0	101·0	103·0	132	110	134	377	—	—	71·8	—	105·4	109·5	16·0	57·3	57·1
—	—	—	114·5	119·0	95·0	136	133	116	385	38·0	30·1	67·0	—	101·7	104·2	20·0	47·0	47·1
535	330	326	119·4	117·5	93·8	140	130	115	385	35·9	28·9	—	136	98·0	—	—	50·0	48·1
—	—	—	116·2	112·0	97·6	134	124	118	376	38·7	—	—	—	—	—	—	56·1	56·1
533	328	325	122·0	121·0	94·0	141	141	116	398	35·4	32·5	—	134	95·6	—	—	55·0	52·1
557	328	325	122·5	114·6	109·0	143	126	134	403	37·9	33·1	69·0	139·5	108·6	109·8	18·0	52·0	52·1
—	307	310	118·3	113·0	—	—	124	—	—	—	—	74·5	133	97·6	—	—	58·0	60·1
530	333	325	116·0	104·0	112·0	132	114	135	381	35·2	30·1	70·8	136·5	100·0	113·2	20·9	50·9	51·1
541	338	334	120·4	128·3	103·0	140	145	123	408	39·5	28·5	—	—	—	—	—	—	—
519	328	321	113·0	109·2	103·0	135	124	124	383	37·3	33·0	70·5	135	99·4	108·2	18·4	53·2	54·1
535	332	325	116·0	121·0	—	133	138	—	—	—	—	70·0	—	—	98·4	17·1	52·1	52·1
538	311	311	114·7	127·0	104·0	136	143	126	405	34·5	28·2	69·0	131	100·2	108·7	20·9	51·1	50·1
555	336	327	123·0	116·8	109·5	142	130	137	409	35·7	32·9	70·6	141	105·0	111·0	18·1	56·3	56·1
511	—	—	110·2	106·3	100·6	125	115	122	362	33·1	30·7	71·1	—	95·4	102·5	17·8	54·5	54·1
538	322	319	119·8	97·0	—	139	104	—	—	—	—	73·4	142	98·2	103·5	21·2	56·7	56·1
520	328	326	112·0	117·6	99·7	130	132·5	122	384·5	34·2	29·0	61·6	129	92·3	100·0	—	49·7	48·1
—	—	—	119·9	117·2	100·2	138	127	120	385	40·3	36·1	73·2	—	—	104·4	16·2	58·0	58·1
506	324	320	110·0	106·2	106·3	126	116	129	371	33·4	30·3	70·0	131	89·3	101·3	—	48·5	50·1
520	319	316	118·0	111·0	88·4	141	122	112	374	—	—	—	—	93·0	—	—	47·1	45·1
545	341	338	123·5	111·5	105·2	144	125	131	400	40·0	32·8	68·3	136	100·7	96·0	—	54·0	55·1
529	326	322	116·9	124·3	104·5	134	139	124	397	36·2	31·6	61·7	132	89·0	98·4	16·5	48·7	46·1
530	315	308	—	122·5	106·0	—	138	128·5	—	30·8	27·3	—	—	—	—	—	—	—
502	323	318	111·5	110·4	95·0	127	126	108	362	40·4	30·9	69·8	131	97·4	97·6	—	55·6	54·1
532	317	316	107·5	120·0	87·5	123	140	104	367	—	—	77·0	142	102·0	107·8	—	59·8	59·1
506	298	302	116·3	106·7	90·7	134	117·5	107	358·5	35·0	32·0	66·5	128	95·0	99·8	—	48·5	50·1
525	324	317	111·2	105·4	104·7	127	117	127·5	371·5	37·0	30·2	66·2	138	99·2	103·0	19·0	51·3	50·1
531	339	337	114·5	118·0	104·8	130	133	120	383	38·6	34·0	68·5	138·5	92·5	99·0	18·2	53·5	54·1
524	312	310	112·6	113·2	105·8	130	124	129	383	36·4	29·5	60·8	128	102·4	95·0	13·2	47·7	47·1
531	—	325	123·8	107·2	116·6	142	115	140	397	35·8	30·0	66·5	132·5	100·0	102·3	—	51·7	51·1
510	—	292	105·0	111·8	99·3	119·5	124·5	120	364	35·5	32·5	67·0	127·5	94·8	96·0	—	47·2	47·1
510	—	307?	112·3	118·1	92·3	129·5	132	110	371·5	34·0	31·0	—	—	—	—	—	—	—
—	—	—	115·2	122·5	—	132	138	—	—	—	—	—	—	—	—	—	—	—
—	—	—	111·0	108·0	99·2	129	120	126	375	34·0	29·9	65·0	131	95·2	101·2	—	48·8	48·1

XII. *Individual Measurements of Male Easter Island Skulls.*

PH	*NHR*	*NHL*	*NH'*	*NB*	*SC*	*SS*	*DC*	*DS*	*DA*	O_1R	$O_1'R$	*Lacr.* O_1R	O_2R	O_1L	$O_1'L$	*Lacr.* O_1L	O_2L	G_1	G_1'
7·3	49·0	49·0	47·6	24·0	8·5	3·1	21·2	8·0	28	40·9	38·2	38·2	35·0	40·9	37·0	37·0	35·1	50·0	46·2
0·4	51·0	51·0	46·8	26·0	7·0	1·5	17·8	8·0	28	41·2	38·3	38·3	36·2	41·9	38·6	38·6	35·3	49·0	44·8
1·0	54·5	54·1	53·0	26·8	10·0	2·9	23·5	11·4	34	43·2	39·1	38·5	32·7	45·0	40·2	38·9	—	57·5	53·0
—	54·1	—	—	26·1	6·2	2·2	21·0	8·9	29·5	—	—	—	—	43·1	40·1	40·1	33·8	—	—
5·7	59·2	59·0	57·7	29·0	7·0	3·5	25·0	10·4	35	45·0	40·9	40·9	36·3	43·7	41·0	39·5	37·4	51·0	47·4
—	55·0	55·0	53·0	27·8	8·0	4·9	24·3	10·4	35·5	45·5	42·2	42·2	34·1	43·5	39·9	39·9	34·0	—	—
8·3	57·5	57·9	57·0	30·0	10·8	4·2	26·3	9·1	34	42·1	39·9	38·5	33·0	42·1	38·4	38·4	33·0	—	—
5·4	53·3	54·5	49·6	26·9	7·0	3·1	23·1	9·1	30·5	44·5	38·9	38·9	35·3	43·0	39·4	39·4	36·3	45·5	43·5
5·1	60·3	60·3	56·8	27·0	8·8	3·2	21·8	10·2	31	43·6	40·0	40·0	35·0	42·3	39·5	39·5	34·5	49·3	43·1
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9·9	55·5	54·0	52·9	28·8	9·0	2·7	25·0	9·2	32·5	43·8	40·1	40·1	31·5	44·0	41·3	39·6	31·5	57·9	52·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	53·0	53·3	51·0	31·0	7·9	4·4	24·0	11·1	37	44·2	40·0	40·0	32·2	—	—	—	33·1	—	—
4·8	50·5	50·4	48·5	24·8	8·4	2·5	22·5	8·9	33	40·9	38·6	38·6	33·9	42·0	38·0	38·0	34·1	47·3	42·9
—	52·8	51·3	48·9	25·3	9·2	3·6	21·3	9·7	31	42·3	38·9	38·9	33·0	41·0	38·2	38·2	33·1	50·0	46·0
7·7	57·4	54·5	53·5	27·8	8·8	3·7	25·0	13·0	38	43·6	40·1	40·1	35·9	43·3	38·9	38·9	34·6	52·8	48·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9·7	58·1	58·1	53·0	27·0	9·0	4·0	26·5	10·3	35·5	44·6	37·9	37·9	34·6	44·3	40·1	40·1	34·3	55·4	51·0
4·8	51·5	—	50·2	29·0	7·5	3·3	23·0	10·8	33	43·0	38·4	38·4	34·0	42·5	39·0	39·0	33·4	51·7	48·5
7·0	58·9	57·2	54·0	29·3	7·5	3·2	22·0 ?	12·5 ?	37 ?	44·0	—	38·5 ?	34·9	43·0	—	36·6	35·3	—	—
—	49·0	49·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6·0	57·3	57·9	54·8	29·5	9·0	4·0	21·3	8·0	28	46·5	44·0	42·1	35·2	—	—	—	35·0	54·1	49·7
0·0	47·0	47·0	45·4	—	7·0	3·4	20·3	8·4	29	41·0	38·1	38·1	33·6	40·9	37·5	37·5	—	52·5	47·2
—	50·0	48·8	48·3	28·3	8·7	3·8	24·2	9·6	34·5	42·5	39·2	39·2	31·5	42·3	40·0	40·0	31·0	—	46·1
—	56·1	56·1	53·2	27·0	8·c	2·6	22·1	10·2	35·5	—	—	—	33·6	43·3	39·5	39·5	34·0	—	—
—	55·0	52·3	53·6	26·3	8·3	3·2	25·0	8·0	30	44·0	40·1	38·3	33·6	45·1	42·0	40·8	35·0	—	—
8·0	52·0	52·0	49·2	28·0	7·2	3·2	23·1	11·9	36	45·0	39·8	39·8	34·9	44·7	41·0	41·0	34·9	52·5	47·8
—	58·0	60·0	—	27·5	6·0	3·5	—	—	—	—	—	—	—	—	—	—	—	—	—
0·9	50·9	51·0	49·2	27·1	3·9	2·0 ?	23·0	9·7	32·5	45·0	41·0	41·0	35·8	45·0	40·1	40·1	35·0	57·6	50·3
—	—	—	—	—	—	—	—	—	—	—	—	—	—	43·0	—	—	37·8	—	—
8·4	53·2	54·6	50·5	26·5	4·5	1·5 ?	21·0	10·9	32	43·1	38·3	38·3	38·0	43·6	39·0	39·0	38·0	51·7	47·1
7·1	52·1	52·9	50·5	—	7·1	2·5	23·3	8·2	31	—	—	—	—	44·5	41·2	41·2	39·5	—	—
0·9	51·1	50·9	47·8	30·4	5·6	3·0 ?	22·0	9·0	32·5	40·0	37·4	37·0	32·5	40·6	37·3	36·1	32·5	53·0	52·0
8·1	56·3	56·0	52·5	28·3	8·0	3·8	24·2	11·8	37	45·2	41·0	40·9	36·1	45·5	42·1	41·0	37·0	56·8	51·1
7·8	54·5	54·5	50·9	27·0	8·2	2·6	22·4	8·9	30·5	43·6	41·8	40·2	34·0	44·5	41·5	39·3	34·3	51·6	46·1
1·2	56·7	56·8	54·5	26·3	9·1	2·5	23·3	10·0	32·5	44·5	42·0	40·4	37·3	47·0	40·9	39·8	37·5	50·5	47·9
—	49·7	48·0	45·0	27·6	9·9	3·5	22·1	8·2	29	41·0	—	37·0	33·5	41·0	—	37·2	34·0	46·7	43·1
6·2	58·0	58·0	55·3	28·2	8·7	4·4	25·3	9·6	31	—	—	—	—	44·6	—	40·7	37·7	—	—
—	48·5	50·0	48·2	29·0	6·8	2·6	21·8	6·8	26	42·0	39·0	—	35·0	42·5	39·0	39·0	35·0	49·0	46·2
—	47·1	45·0	44·4	25·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	54·0	55·8	51·0	28·4	7·2	1·0	24·0	7·0	31·5	42·5	—	—	32·1	42·2	—	—	32·0	—	—
6·5	48·7	46·0	44·2	25·6	7·4	2·3	21·3	8·8	29	41·4	38·8	38·8	33·8	41·4	38·2	38·2	34·1	48·3	44·7
—	—	—	—	—	—	—	—	—	—	—	—	—	—	45·1	—	—	33·7	—	—
—	55·6	54·5	51·5	26·3	7·0	2·2	21·6	7·0	28·5	43·2	—	40·0	34·8	43·7	41·0	39·5	33·2	48·0	43·6
—	59·8	59·8	55·0	30·0	9·0	5·8	—	—	—	45·4	—	—	35·2	45·4	—	—	35·5	50·8	47·0
—	48·5	50·0	48·4 ?	26·1	6·9	2·5	21·0	10·1	29	42·7	—	38·8	34·5	42·5	—	38·8	35·5	—	46·0
9·0	51·3	50·9	46·6	26·1	—	—	23·2	10·9	32	43·8	—	39·6	33·5	44·1	—	39·0	34·2	48·8	45·0
8·2	53·5	54·3	50·0	25·7	—	—	21·0	9·0	28	41·2	—	37·5	34·0	41·5	39·8	37·5	33·2	48·0	44·3
3·2	47·7	47·7	47·5	23·0	9·0	2·0	22·0	8·0	26·5	42·3	—	39·2	35·0	44·8	—	41·1	34·0	47·1	44·5
—	51·7	51·8	48·6	27·0	7·1	3·5	19·8	9·0	—	45·5	41·8	40·1	34·8	46·5	41·2	40·5	34·9	51·5	46·8
—	47·2	47·9	46·0	24·6	7·0	2·1	18·9	12·3	—	43·6	40·2	39·0	34·8	42·5	39·2	38·5	35·0	40·2	38·0
—	—	—	—	—	—	—	21·6	9·0	—	45·0	41·6	40·3	34·6	—	—	—	—	—	—
—	—	—	—	—	7·2	3·2	21·0	7·0	—	—	—	—	—	—	—	—	—	—	—
—	48·8	48·8	46·5	25·0	7·8	3·1	19·8	8·6	—	42·6	40·4	40·4	36·3	43·0	39·6	39·6	34·5	43·6	42·0

s.

r.	O_2R	O_1L	$O_1'L$	Lacr. O_1L	O_2L	G_1	G_1'	G_2	EH	Alveolar $P\angle$	Prosthion $P\angle$	$N\angle$	$A\angle$	$B\angle$	Oc. I.	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{B}{H'}$	100 $\frac{(B-L}{L}$
2	35·0	40·9	37·0	37·0	35·1	50·0	46·2	41·2	15·1	86°·9	84°·0	67°·4	75°·8	36°·8	60·5	69·1	72·9	94·9	– 3·
3	36·2	41·9	38·6	38·6	35·3	49·0	44·8	—	—	86°·0	83°·8	68°·8	74°·1	37°·1	61·7	69·9	77·9	89·8	– 7·
5	32·7	45·0	40·2	38·9	—	57·5	53·0	42·1	12·3	—	—	—	—	—	—	—	—	—	—
	—	43·1	40·1	40·1	33·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9	36·3	43·7	41·0	39·5	37·4	51·0	47·4	39·8	12·0	86°·1	83°·7	66°·5	74°·6	38°·9	60·6	68·8	—	—	—
2	34·1	43·5	39·9	39·9	34·0	—	—	—	—	—	—	—	—	—	63·2	78·5	78·0	100·7	+ 0·6
5	33·0	42·1	38·4	38·4	33·0	—	—	39·8	14·9	92°·8	90°·6	61°·9	72°·4	45°·7	59·1	75·8	75·3	100·7	+ 0·5
9	35·3	43·0	39·4	39·4	36·3	45·5	43·5	—	—	88°·2	86°·0	64°·8	78°·4	36°·8	60·8	67·4	78·2	86·2	– 10·8
0	35·0	42·3	39·5	39·5	34·5	49·3	43·1	37·2	10·2	100°·1	98°·0	59°·0	82°·8	38°·2	60·0	71·6	75·2	95·2	– 3·6
1	31·5	44·0	41·3	39·6	31·5	57·9	52·0	39·0	10·0	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	87°·3	85°·3	68°·4	72°·8	38°·8	56·2	65·4	72·6	90·1	– 7·2
0	32·2	—	—	—	33·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6	33·9	42·0	38·0	38·0	34·1	47·3	42·9	35·4	10·4	89°·2	85°·1	67°·7	74°·2	38°·1	—	68·5	74·4	92·0	– 5·9
9	33·0	41·0	38·2	38·2	33·1	50·0	46·0	38·0	8·6	89°·0	86°·0	65°·6	78°·9	35°·5	59·6	—	72·7	—	—
1	35·9	43·3	38·9	38·9	34·6	52·8	48·0	43·0	11·7	91°·0	86°·2	—	—	—	60·2	70·4	76·7	91·7	– 6·3
	—	—	—	—	—	—	—	—	—	86°·3	84°·0	68°·9	71°·4	39°·7	—	69·3	73·4	94·5	– 4·1
9	34·6	44·3	40·1	40·1	34·3	55·4	51·0	—	—	—	—	—	—	—	58·6	66·1	—	—	—
4	34·0	42·5	39·0	39·0	33·4	51·7	48·5	—	—	86°·2	84°·3	67°·0	73°·8	39°·2	58·1	72·0	74·6	96·5	– 2·6
5?	34·9	43·0	—	36·6	35·3	—	—	—	—	82°·8	80°·0	70°·9	73°·0	36°·1	59·2	70·7	75·3	93·8	– 4·6
	—	—	—	—	—	—	—	—	—	88°·0	86°·0	67°·4	73°·1	39°·5	59·4	69·9	74·9	93·4	– 5·0
1	35·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
1	33·6	40·9	37·5	37·5	35·0	54·1	49·7	—	—	89°·0	86°·3	65°·2	78°·4	36°·4	55·2	65·1	70·9	91·7	– 5·9
2	31·5	42·3	40·0	40·0	—	52·5	47·2	38·1	13·6	86°·1	83°·0	69°·9	73°·0	37°·1	58·6	—	71·5	—	—
	33·6	43·3	39·5	39·5	31·0	—	46·1	—	—	83°·0	81°·5	—	—	—	58·3	69·4	75·3	92·1	– 6·0
3	33·6	45·1	42·0	40·8	34·0	—	—	—	—	—	—	—	—	—	59·4	—	75·1	—	—
8	34·9	44·7	41·0	41·0	35·0	—	—	—	—	—	—	—	—	—	57·8	—	72·5	—	—
	—	—	—	—	34·9	52·5	47·8	44·9	14·5	87°·5	85°·0	67°·4	77°·3	35°·3	58·1	64·8	71·8	90·2	– 7·0
0	35·8	45·0	40·1	40·1	35·0	57·6	50·3	45·5	13·3	91°·0	—	—	—	—	—	—	—	—	—
	—	43·0	—	—	37·8	—	—	—	—	81°·2	78°·4	74°·1	69°·0	36°·9	59·6	71·6	76·3	93·8	– 4·7
3	38·0	43·6	39·0	39·0	38·0	51·7	47·1	38·2	11·7	—	—	—	—	—	60·5	65·5	73·5	89·1	– 8·0
	—	44·5	41·2	41·2	39·5	—	—	—	—	88°·0	86°·0	68°·5	74°·3	37°·2	59·7	65·8	75·3	87·4	– 9·5
0	32·5	40·6	37·3	36·1	32·5	53·0	52·0	42·6	11·5	92°·8	90°·3	60°·2	81°·7	38°·1	—	70·1	76·3	91·8	– 6·3
9	36·1	45·5	42·1	41·0	37·0	56·8	51·1	—	—	84°·1	81°·1	69°·9	73°·8	36°·3	59·2	64·3	72·6	88·6	– 8·3
2	34·0	44·5	41·5	39·3	34·3	51·6	46·1	39·9	11·4	84°·9	80°·8	67°·3	76°·8	35°·9	57·0	67·7	76·2	88·9	– 8·4
4	37·3	47·0	40·9	39·8	37·5	50·5	47·9	—	—	94°·0	91°·2	61°·6	80°·9	37°·5	59·1	—	76·6	—	—
0	33·5	41·0	—	37·2	34·0	46·7	43·1	—	—	90°·5	86°·2	62°·0	79°·2	38°·8	—	67·5	73·4	92·0	– 5·8
	—	44·6	—	40·7	33·7	—	—	—	—	89°·6	87°·8	66°·1	79°·8	34°·1	58·4	70·4	74·3	94·7	– 4·0
	35·0	42·5	39·0	39·0	35·0	49·0	46·2	—	—	93°·8	91°·7	61°·3	80°·8	37°·9	60·2	67·8	74·6	90·8	– 6·9
	—	—	—	—	—	—	—	—	—	86°·0	84°·0	67°·6	72°·7	39°·7	59·0	73·2	76·5	95·6	– 3·3
	32·1	42·2	—	—	32·0	—	—	—	—	—	—	—	—	—	56·3	72·3	—	—	—
8	33·8	41·4	38·2	38·2	34·1	48·3	44·7	—	—	100°·5	97°·3	58°·4	84°·5	37°·1	57·3	70·7	73·2	96·6	– 2·5
0	—	45·1	—	—	33·7	—	—	—	—	89°·9	85°·7	67°·7	77°·2	35°·1	61·0	67·6	72·8	92·9	– 5·2
	34·8	43·7	41·0	39·5	33·2	48·0	43·6	42·5	10·0	—	—	—	—	—	59·2	70·4	71·5	98·5	– 1·0
8	35·2	45·4	—	—	33·7	—	47·0	—	—	92°·4	89°·6	64°·1	75°·9	40°·0	59·2	70·4	71·5	98·5	– 1·0
6	34·5	42·5	—	38·8	35·5	50·8	—	—	—	86°·0	82°·3	62°·8	77°·9	39°·3	60·9	70·4	76·7	90·6	– 7·2
5	33·5	44·1	—	39·0	35·5	—	46·0	—	—	87°·8	85°·0	66°·0	76°·6	37°·4	61·7	68·7	76·4	92·1	– 6·0
2	34·0	41·5	39·8	37·5	34·2	48·8	45·0	—	—	92°·0	88°·1	64°·9	79°·6	35°·5	58·8	70·6	73·8	93·0	– 5·2
1	35·0	44·8	—	41·1	33·2	48·0	44·3	37·8	10·2	93°·0	88°·7	61°·5	81°·3	37°·2	65·7	75·5	81·9	95·4	– 3·4
0	34·8	46·5	41·2	40·5	34·0	47·1	44·5	40·8	11·2	88°·1	84°·7	65°·9	78°·8	35°·3	58·7	68·8	70·4	92·2	– 6·4
3	34·8	42·5	39·2	38·5	34·9	51·5	46·8	35·0	—	—	—	64°·3	79°·6	36°·1	59·9	66·3	79·3	97·8	– 1·6
	34·6	—	—	—	35·0	40·2	38·0	36·0	—	—	—	61°·3	80°·7	38°·0	59·4	67·7	74·7	83·7	– 13·0
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	60·6	66·7	74·5	90·5	– 7·1
4	36·3	43·0	39·6	39·6	34·5	43·6	42·0	37·1	—	—	—	67°·7	75°·8	36°·5	—	66·4	—	89·5	– 7·8
															56·2	—	71·0	—	—

100 $\frac{B}{H'}$	100 $\frac{(B-H')}{L}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH}, R$	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{O_1}, L$	100 $\frac{O_2}{O_1'}, R$	100 $\frac{G_2}{G_1}$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{EH}{G_2}$	100 $\frac{fmb}{fml}$
94·9	− 3·7	66·8	50·4	49·0	85·6	85·8	91·6	82·4	37·7	36·5	36·7	77·1
89·8	− 7·9	77·3	55·6	51·0	87·9	84·8	94·5	—	44·9	21·4	—	80·5
—	—	74·9	50·6	49·2	75·7	—	83·6	73·2	48·5	29·0	29·3	—
—	—	—	—	48·2	—	78·4	—	—	42·4	35·5	—	—
—	—	69·9	50·2	49·0	80·7	85·6	88·8	78·0	41·6	50·0	30·2	84·4
100·7	+ 0·6	—	52·5	50·5	74·9	78·2	80·8	—	42·8	61·3	—	83·1
100·7	+ 0·5	76·0	52·6	52·2	78·4	78·4	82·7	—	34·6	38·9	37·4	84·7
86·2	− 10·8	70·1	54·2	50·5	79·3	84·4	90·7	—	39·4	44·3	—	—
95·2	− 3·6	75·6	47·5	44·8	80·3	81·6	87·5	75·5	46·8	36·4	27·4	84·6
—	—	—	—	—	—	—	—	—	—	—	—	—
90·1	− 7·2	73·6	54·4	51·9	71·9	71·6	78·6	67·4	36·8	30·0	25·6	—
—	—	—	—	—	—	—	—	—	—	—	—	—
92·0	− 5·9	77·2	60·8	58·5	72·9	—	80·5	—	46·3	55·7	—	—
—	—	68·2	51·1	49·1	82·9	81·2	87·8	74·8	39·6	29·8	29·4	—
91·7	− 6·3	—	51·7	47·9	78·0	80·7	84·8	76·0	45·5	39·1	22·6	83·7
94·5	− 4·1	72·5	52·0	48·4	82·3	79·9	89·5	81·4	52·0	42·0	27·2	—
—	—	—	—	—	—	—	—	—	—	—	—	—
96·5	− 2·6	82·2	50·9	46·5	77·6	77·4	91·3	—	38·9	44·4	—	76·9
93·8	− 4·6	65·8	57·8	56·3	79·1	78·6	88·5	—	47·0	44·0	—	80·6
93·4	− 5·0	73·2	54·3	49·7	79·3	82·1	—	—	—	42·7	—	83·4
—	—	—	—	—	—	—	—	—	—	—	—	—
91·7	− 5·9	68·1	53·8	51·5	75·9	—	80·0	—	37·6	44·4	—	—
—	—	65·9	—	—	82·0	—	88·2	72·6	41·4	48·6	35·7	79·2
92·1	− 6·0	—	58·6	56·6	74·1	73·3	80·4	—	39·7	43·7	—	80·5
—	—	—	50·8	48·1	—	78·5	—	—	46·2	32·5	—	—
—	—	—	49·1	47·8	76·4	77·6	83·8	—	34·8	38·6	—	91·8
90·2	− 7·0	63·5	56·9	53·8	77·6	78·1	87·7	88·5	51·5	44·4	32·3	87·3
—	—	76·3	—	47·4	—	—	—	—	—	58·3	—	—
93·8	− 4·7	70·8	55·1	53·2	79·6	77·8	87·3	79·0	42·2	51·3 ?	29·2	85·5
89·1	− 8·0	—	—	—	—	87·9	—	—	—	—	—	72·2
87·4	− 9·5	70·9	52·5	49·8	88·2	87·2	99·2	73·9	51·9	33·3 ?	30·6	88·5
91·8	− 6·3	—	—	—	—	88·8	—	—	35·2	35·2	—	—
88·6	− 8·3	69·0	63·6	59·5	81·3	80·0	86·9	80·4	40·9	53·6 ?	27·0	81·7
88·9	− 8·4	67·2	53·9	50·3	79·9	81·3	86·8	—	48·8	47·5	—	92·2
—	—	74·5	53·0	49·5	78·0	77·1	81·3	77·3	39·7	31·7	28·6	92·7
92·0	− 5·8	74·7	48·3	46·4	83·8	79·8	88·8	—	43·1	27·5	—	—
94·7	− 4·0	66·7	61·3	55·5	81·7	82·9	—	—	37·1	35·4	—	84·8
90·8	− 6·9	—	51·0	48·6	—	75·6	—	—	37·9	50·6	—	89·6
95·6	− 3·3	78·4	60·2	59·8	83·6	82·4	89·7	—	31·2	38·2	—	90·7
—	—	—	57·0	53·7	—	—	—	—	—	—	—	—
96·6	− 2·5	67·8	55·7	52·6	75·5	75·8	—	—	29·2	13·9	—	82·0
92·9	− 5·2	79·3	57·9	52·6	81·6	82·4	87·1	—	41·3	31·1	—	87·3
98·5	− 1·0	—	—	—	—	74·7	—	—	—	—	—	88·6
90·6	− 7·2	71·7	51·1	47·3	80·6	76·0	—	88·5	32·4	31·5	23·5	76·5
92·1	− 6·0	75·5	54·5	50·2	77·5	78·2	—	—	—	64·4	—	—
93·0	− 5·2	70·0	53·9 ?	53·8	80·8	83·5	—	—	48·1	36·2	—	91·4
95·4	− 3·4	66·7	56·0	50·9	76·5	77·6	—	—	47·0	—	—	81·6
92·2	− 6·4	74·1	51·4	48·0	82·5	80·0	—	78·8	42·9	—	27·0	88·1
97·8	− 1·6	59·4	48·4	48·2	82·7	75·9	—	86·6	36·4	22·2	27·5	81·0
83·7	− 13·0	66·5	55·6	52·2	76·5	75·1	83·3	68·0	45·5	49·3	—	83·8
90·5	− 7·1	70·7	53·5	52·1	79·8	82·4	86·6	89·6	65·1	30·0	—	91·5
89·5	− 7·8	—	—	—	76·9	—	83·2	—	41·7	—	—	91·2
—	—	—	—	—	—	—	—	—	33·3	44·4	—	—
—	—	68·3	53·8	51·2	85·2	80·2	89·9	85·1	43·4	39·7	—	87·9

		C	L	F	B	H′	H	OH	Bregmatic OH	LB	B′	U	Glabella U	Q′	Br ma Q
FEMALES.	R.C.S. 806·03	1305	181	181	120	139	140	119	117	103·5	87·0	491	493	—	—
	·05	—	177·5	177	126·5	136·5	138	120·5	119	103·5	93·5	486	492	320	31
	·07	1320	180	180·5	126	145	146	122	119	113	94·6	502	505	315	30
	·13	—	185	186	—	143	141	124	127	104	94·3	—	—	324	32
	·14	1335	182·5	181·5	133	138	144	119	117	100	87·2	495	502	311	30
	·18	1290	184·5	187	125	136	136·5	119·5	119·5	99	98·2	510	512	312	30
	·20	1400	183	182	129	141·5	145	123·5	121	101	91·4	502	507	319	31
	·22	—	186	185	—	139·5	—	—	—	108	—	—	—	—	—
	·24	—	175	175	126·5	131	135	116	113·5	100	—	487	491	306	29
	·25	1305	182	181	130	138	—	—	—	103·5	—	497	504	—	30
	·42	1400	185	186	137	141	143·5	124	122·5	102	91·6	513	513	325	32
	·43	—	177·5	179	—	131	133	116·5	115	100	—	480?	485?	302?	29
	·44	—	175·5	177	—	—	—	—	—	—	—	—	—	—	—
	·48	—	184	186	125	134	—	—	—	104	97·4	501	504	—	30
	·55	1210	179·5	180	129	136	136·5	117·5	117	105·5	94·0	494	499	314	30
	·57	1365	177·5	178	126	142	143	120	119	106·5	91·0	489	495	315	30
	Leiden 191	—	181·5	182	134	135	134	117	117	107	87·5	497	499	317	31
	193	—	176	175·5	136	139	140	116·5	116	100·6	93·6	492	496	309	31
	194	—	176	176·5	131	137	135	110·5	110·5	103·9	88·6	495	493	299	30
	197	—	179	179	136	131	129	119	118·5	97·4	93·1	502	506	322	31
	206	—	181·5	181·5	136	138	138	120	120	97·8	89·0	509	511	—	31
	B.M. 5	—	183·5	183·5	124·5	129·5	—	—	—	99·5	84·5	492·5	496	—	29
	8	—	175·5	176	134·5	137	—	—	—	101·1	88·0	486	490	—	31
	7	—	176·5	178·5	127	135	—	—	—	99·7	90·2	492	496	—	31
	13	—	177	176	—	—	—	—	—	—	89·2	—	—	—	—
JUVENILES	R.C.S. 806·04	—	188	188·5	133	138	136·5	121·5	122·5	107	94·3	512	520	323	31
	·09	—	168	170	131	125	126·5	113	112	91	88·0	468	—	312	29
	·12	—	168	170	134	131	130	121·5	122	100·5	95·2	489	490	328	31
	·23	—	183·5	183	132·5	135	137·5	117·5	116·5	104·5	92·0	564	510	310	31
	·49	—	189·5	190·5	131·5	138	138	117	117	103	91·0	512	514	312	31
	·54	—	180·5	181·5	126·5	136·5	138	118·5	117·5	102	89·0	492	497	314	31
	B.M. 2	—	174	176·5	126	133	—	—	—	98	89·0	485	487	—	29
	4	—	180	180	139·5	—	—	—	—	—	86·0	503	505	—	31

Flabella U	*Q'*	*Bregmatic Q'*	S_1'	S_2'	S_3'	S_1	S_2	S_3	*S*	*fml*	*fmb*	*G'H*
493	—	—	107·8	112·2	92·1	122	127	109	358	—	—	67·3
492	320	310	108·0	99·4	107·7	124·5	107·5	126	358	33·2	30·0	63·9
505	315	309	109·8	101·0	109·3	124	110	130	364	—	33·2	68·5
—	324	323	115·0	115·2	99·2	136	127	121	384	34·1	29·1	63·0
502	311	301	110·0	117·0	96·0	125	134	116	375	—	30·1	—
512	312	302	105·5	126·0	—	127	139·5	—	—	—	27·3	58·9
507	319	311	113·8	123·2	97·7	128	139	115	382	33·5	28·1	—
—	—	—	105·0	121·0	100·0	120	137	115	372	35·0	30·9	66·0
491	306	295	102·5	101·0	103·3	118	110	124	352	35·5	29·6	—
504	—	306	110·3	115·7	98·5	125	130	116	371	35·0	29·9	—
513	325	320	116·7	114·3	99·3	133	128	119	380	35·2	28·5	—
485?	302?	297?	102·2	112·7	97·0	117	126	119	362	32·8	29·0	—
—	—	—	109·5	112·0	—	124	126	—	—	—	—	—
504	—	302	113·5	118·0	—	132	135	—	—	—	—	—
499	314	307	112·0	108·7	89·5	130	122	111	363	33·2	27·9	67·2
495	315	306	110·6	103·0	100·5	126	112	116	354	39·4	28·5	60·1
499	317	316	112·0	112·8	94·1	127	127	112	366	32·6	30·3	62·5
496	309	511	112·4	113·0	99·8	125	125	113	363	35·0	29·0	61·6
493	299	300	106·8	99·2	98·4	123	108	121	351	33·0	33·2?	64·2
506	322	325	114·1	108·3	99·2	134	119·5	120·5	374	35·0	—	58·4
511	—	322	116·3	101·0	112·7	136	110	141	387	33·2	31·0	60·6
496	—	292	109·2	112·2	87·0	128·5	127	109	365	35·9	29·3	58·1
490	—	310	106·0	102·0	104·0	124	113	125·5	362	32·6	27·5	—
496	—	302	116·0	112·5	88·3	135	128	106	370	31·4	27·2	61·5
—	—	—	112·5	112·0	—	128	125	—	—	—	—	—
520	323	319	114·0	109·5	105·3	133	121	127	381	35·2	31·6	—
—	312	299	104·0	108·0	80·8	124	124	99	347	31·5	26·8	—
490	328	319	110·0	107·0	89·0	131	124	105	360	25·5	20·9	57·2
510	310	305	113·0	110·0	97·0	132	123	114	369	34·9	29·3	57·4
514	312	308	111·3	118·0	88·0	129	132	113	374	40·0	29·6	56·4
497	314	307	110·0	110·0	93·1	128·5	121	113	363	35·0	28·2	63·7
487	—	299	103·0	111·5	92·8	119	124	113	356	32·1	26·8	53·8
505	—	314	111·0	108·7	99·6	129	120	122	371	—	28·0	67·0

TABLE XIII. *Individual Measurements of Female and Juveni*

S_1	S_2	S_3	S	*fml*	*fmb*	*G'H*	*J*	*GB*	*GL*	*PH*	*NHR*	*NHL*	*NH'*	*NB*	*SC*	*SS*	*DC*	*DS*	*DA*
22	127	109	358	—	—	67·3	—	—	94·0	16·7	52·4	53·0	49·2	21·2	7·0	2·1	21·5	7·5	28
24·5	107·5	126	358	33·2	30·0	63·9	—	—	95·0	15·6	47·9	47·8	47·0	—	6·5	2·1	25·1	7·8	33
24	110	130	364	—	33·2	68·5	127	92·0	104·0	19·0	53·7	53·0	50·5	27·3	7·3	2·0	23·0	8·0	30
36	127	121	384	34·1	29·1	63·0	—	86·0	92·2	15·2	50·8	49·0	47·5	24·0	9·2	3·0	22·3	9·0	30·5
25	134	116	375	—	30·1	—	118·5	90·9	—	—	47·0	48·9	41·2	25·9	6·5	2·5	21·0	7·0	28
27	139·5	—	—	—	27·3	58·9	125	94·0	91·5	11·9	48·2	48·2	47·2	26·7	11·0	2·8	26·0	9·0	33
28	139	115	382	33·5	28·1	—	125·5	93·6	—	—	46·6	48·0	42·6	25·0	7·0	2·3	21·6	6·5	27·5
20	137	115	372	35·0	30·9	66·0	—	—	107·0	17·9	48·3	47·7	46·6	24·8	8·5	2·5	24·1	11·5	36·5
18	110	124	352	35·5	29·6	—	—	—	—	—	45·[illegible]	46·3	43·1	25·0	7·2	2·1	21·8	5·0	24·5
25	130	116	371	35·0	29·9	—	—	—	—	—	—	—	—	—	9·2	2·0	25·0	10·0	33
33	128	119	380	35·2	28·5	—	126·5	93·0	—	—	49·0	48·2	46·0	25·5	6·9	2·1	23·0	7·0	30
17	126	119	362	32·8	29·0	—	—	—	—	—	45·9	46·4	43·0	24·8	8·1	3·0	18·7	7·6	28
24	126	—	—	—	—	—	—	—	—	—	—	—	—	—	10·3	2·8	23·0	9·1	31
32	135	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
30	122	111	363	33·2	27·9	67·2	123	95·0	97·0	—	50·5	51·2	49·4	25·8	9·6	4·0	22·7	9·4	32
26	112	116	354	39·4	28·5	60·1	126·5	90·2	97·2	12·4	49·0	49·1	47·0	26·2	8·8	3·0	21·0	9·2	33
27	127	112	366	32·6	30·3	62·5	123·5	88·0	99·8	15·0	50·0	48·0	46·5	25·3	10·2	3·8	21·0	8·6	27·5
25	125	113	363	35·0	29·0	61·6	120	99·5	92·3	15·2	46·8	47·5	45·9	24·0	8·0	2·4	—	—	—
23	108	121	351	33·0	33·2 ?	64·2	126	87·0	99·0	15·5	48·7	49·6	47·7	24·0	7·2	3·0	—	—	—
34	119·5	120·5	374	35·0	—	58·4	123	—	98·0	—	44·7	42·5	42·0	26·5	7·9	2·5	—	—	—
36	110	141	387	33·2	31·0	60·6	—	—	91·0	17·0	47·2	47·6	43·0	25·3	—	—	20·3	7·0	25
28·5	127	109	365	35·9	29·3	58·1	—	—	96·0	—	47·5	47·2	46·8	26·6	5·5	—	19·6	9·5	—
24	113	125·5	362	32·6	27·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—
35	128	106	370	31·4	27·2	61·5	126	98·5	95·5	—	47·6	50·0	47·4	28·0	6·9	1·2	21·1	9·5	—
28	125	—	—	—	—	—	—	—	—	—	—	—	—	—	8·8	1·3	20·5	7·8	—
33	121	127	381	35·2	31·6	—	—	—	—	—	—	—	—	—	7·8	2·5	21·0	8·2	29·5
24	124	99	347	31·5	26·8	—	106	79·2	—	—	47·0	49·1	47·0	23·3	7·0	1·8	19·2	7·0	26
31	124	105	360	25·5	20·9	57·2	124	91·3	94·8	15·7	39·0	39·0	37·9	22·0	8·3	1·5	21·6	7·1	29·5
32	123	114	369	34·9	29·3	57·4	126	89·6	95·7	13·4	45·0	43·8	42·4	26·5	7·2	2·2	20·0	9·0	30·5
29	132	113	374	40·0	29·6	56·4	—	92·6	95·0	13·5	45·0	46·2	43·4	25·9	9·4	2·6	22·3	11·2	34·5
28·5	121	113	363	35·0	28·2	63·7	116·5	90·0	94·3	17·2	47·5	46·1	45·7	22·9	8·8	2·4	19·0	6·4	26
19	124	113	356	32·1	26·8	53·8	111·8	82·5	90·6	—	43·4	43·3	42·7	25·1	4·5	—	18·0	7·1	—
29	120	122	371	—	28·0	67·0	122·0	86·1	—	—	48·4	49·4	46·0	22·3	6·4	2·3	17·9	9·0	—

and Juvenile Easter Island Skulls.

DS	*DA*	O_1R	$O_1'R$	*Lacr.* O_1R	O_2R	O_1L	$O_1'L$	*Lacr.* O_1L	O_2L	G_1	G_1'	G_2	*EH*	*Alveolar* $P\angle$	*Prosthion* $P\angle$
7·5	28	41·0	37·8	37·4	34·0	41·0	38·3	37·8	34·0	44·0	—	—	—	99°·4	87°·6
7·8	33	—	—	—	—	42·0	37·4	37·4	34·5	43·1	39·8	35·8	7·7	92°·0	89°·6
8·0	30	41·4	39·8	37·3	31·8	42·0	38·0	36·8	32·8	49·0	46·1	—	—	90°·0	86°·1
9·0	30·5	40·3	37·0	37·0	33·8	40·1	37·0	37·0	34·0	46·0	40·8	37·8	9·7	91°·7	91°·0
7·0	28	40·0	38·2	36·9	33·7	39·9	36·2	35·3	33·9	—	—	—	—	—	—
9·0	33	41·1	37·5	37·5	35·0	42·2	37·4	37·4	36·0	46·6	44·0	—	—	87°·9	83°·8
6·5	27·5	41·5	38·4	38·0	35·8	41·2	38·2	37·0	35·8	—	—	—	—	92°·0	89°·6
11·5	36·5	44·4	40·3	40·3	35·0	—	—	—	—	52·0	48·1	—	—	—	—
5·0	24·5	41·9	39·0	39·0	33·3	—	—	—	—	—	—	—	—	—	—
10·0	33	41·7	36·8	36·8	33·2	—	—	—	—	—	—	—	—	—	—
7·0	30	42·3	39·0	38·0	34·0	42·7	39·0	38·2	34·0	—	—	—	—	92°·0	90°·2
7·6	28	—	—	—	32·9	38·5	34·3	34·3	32·2	—	—	—	—	—	—
9·1	31	41·3	38·4	37·6	32·0	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9·4	32	42·0	38·6	38·6	32·6	44·0	39·0	39·0	33·0	50·2	46·8	37·8	8·7	90°·5	—
9·2	33	41·0	37·0	37·0	36·0	41·5	37·0	37·0	36·0	43·2	40·5	35·8	6·4	86°·0	83°·8
8·6	27·5	41·0	—	36·9	34·0	39·0	—	35·9	33·6	48·3	45·0	39·0	9·2	90°·5	87°·7
—	—	42·2	—	—	33·5	40·5	—	—	33·5	44·5	41·5	39·8	9·0	87°·2	85°·5
—	—	41·0	—	—	33·3	40·2	—	36·6	33·0	48·0	44·5	34·2	11·6	82°·7	79°·8
—	—	40·1	—	—	31·3	41·0	—	—	31·1	52·5	47·0	—	—	87°·6	84°·5
7·0	25	40·0	38·6	37·0	33·5	40·4	38·6	37·0	33·6	43·0	40·0	37·0	—	90°·5	—
9·5	—	43·0	38·9	37·5	33·6	43·6	39·6	38·0	33·7	50·0	45·5	34·5	—	—	—
—	—	—	—	—	—	41·1	—	—	35·7	—	—	—	—	—	—
9·5	—	42·5	39·2	39·2	33·0	41·8	38·0	38·0	34·3	—	—	—	—	—	—
7·8	—	43·0	39·7	39·7	36·0	—	—	—	—	—	—	—	—	—	—
8·2	29·5	41·2	37·3	37·3	—	—	36·5	36·5	—	47·8	43·4	40·2	10·2	85°·7	83°·0
7·0	26	36·2	34·0	34·0	31·9	37·1	34·2	34·2	32·0	36·7	33·7	—	—	90°·5	85°·2
7·1	29·5	40·7	37·0	37·0	33·0	40·9	37·4	37·4	33·0	—	44·2	38·7	11·0	92°·8	91°·0
9·0	30·5	40·6	38·9	37·8	32·8	40·3	38·0	36·9	32·8	48·5	46·0	35·0	6·8	90°·0	88°·2
11·2	34·5	40·0	36·0	36·0	35·8	41·0	36·9	36·9	35·8	48·0	42·6	40·8	7·8	87°·1	85°·0
6·4	26	41·3	39·8	38·3	34·0	40·6	38·4	38·1	33·8	45·2	42·1	35·0	5·2	89°·8	86°·9
7·1	—	40·8	36·7	36·7	33·5	39·4	36·8	36·8	34·0	45·0	41·4	32·7	—	—	—
9·0	—	41·1	38·2	38·2	36·3	40·5	36·9	36·9	35·9	—	43·3	—	—	—	—

$O_1'L$	*Lacr.* O_1L	O_2L	G_1	G_1'	G_2	*EH*	*Alveolar* $P\angle$	*Prosthion* $P\angle$	$N\angle$	$A\angle$	$B\angle$	*Oc. I.*	$100\frac{B}{L}$	$100\frac{H'}{L}$	$100\frac{B}{H'}$	$100\frac{(B-H')}{L}$	$100\frac{G'H}{GB}$	$100\frac{NB}{NH'}$	$100\frac{NB}{NH}$
38·3	37·8	34·0	44·0	—	—	—	99°·4	87°·6	62°·6	78°·0	39°·4	61·4	66·3	76·8	86·3	−10·5	—	43·1	40
37·4	37·4	34·5 ?	43·1	39·8	35·8	7·7	92°·0	89°·6	64°·1	78°·8	37°·1	62·7	71·3	76·9	92·7	− 5·6	—	—	—
38·0	36·8	32·8	49·0	46·1	—	—	90°·0	86°·1	64°·5	79°·2	36°·3	60·9	70·0	80·6	86·9	−10·6	74·5	54·1	50
37·0	37·0	34·0	46·0	40·8	37·8	9·7	91°·7	91°·0	61°·3	81°·9	36°·8	58·6	—	77·3	—	—	73·3	50·5	47
36·2	35·3	33·9	—	—	—	—	—	—	—	—	—	59·4	72·9	75·6	96·4	− 2·7	—	55·1	55
37·4	37·4	36·0	46·6	44·0	—	—	87°·9	83°·8	65°·1	79°·3	35°·6	—	67·7	73·7	91·9	− 6·0	62·7	56·6	55
38·2	37·0	35·8	—	—	—	—	92°·0	89°·6	—	—	—	62·0	70·5	77·3	90·2	− 6·8	—	58·7	53
—	—	—	52·0	48·1	—	—	—	—	71°·2	73°·1	35°·7	65·0	—	75·0	—	—	—	53·2	51
—	—	—	—	—	—	—	—	—	—	—	—	60·0	72·3	74·9	96·6	− 2·6	—	58·0	54
—	—	—	—	—	—	—	—	—	—	—	—	61·9	71·4	75·8	94·2	− 4·4	—	—	—
39·0	38·2	34·0	—	—	—	—	92°·0	90°·2	—	—	—	60·2	74·1	76·2	97·2	− 2·2	—	55·4	52
34·3	34·3	32·2	—	—	—	—	—	—	—	—	—	58·2	—	73·8	—	—	—	57·7	54
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	67·9	72·8	93·3	− 4·9	—	—	—
39·0	39·0	33·0	50·2	46·8	37·8	8·7	90°·5	—	63°·9	77°·8	38°·3	57·5	71·9	75·8	94·9	− 4·0	70·7	52·2	51
37·0	37·0	36·0	43·2	40·5	35·8	6·4	86°·0	83°·8	64°·7	81°·4	33°·9	64·5	71·0	80·0	88·7	− 9·0	66·6	55·7	53
—	35·9	33·6	48·3	45·0	39·0	9·2	90°·5	87°·7	66°·2	79°·0	34°·8	60·8	73·8	74·4	99·3	− 0·6	71·0	54·4	50
—	—	33·5	44·5	41·5	39·8	9·0	87°·2	85°·5	64°·2	79°·0	36°·8	67·7	77·3	79·0	97·8	− 1·7	61·9	52·3	51
—	36·6	33·0	48·0	44·5	34·2	11·6	82°·7	79°·8	67°·4	75°·8	36°·8	58·1	74·4	77·8	95·6	− 3·4	73·8	50·3	49
—	—	31·1	52·5	47·0	—	—	87°·6	84°·5	73°·2	72°·2	34°·6	59·0	76·0	73·2	103·8	+ 2·8	—	63·1	59
38·6	37·0	33·6	43·0	40·0	37·0	—	90°·5	—	65°·4	77°·5	37°·1	57·0	74·9	76·0	98·6	− 1·1	—	58·8	53
39·6	38·0	33·7	50·0	45·5	34·5	—	—	—	69°·7	75°·8	34°·5	56·9	67·8	70·6	96·1	− 2·7	—	56·8	56
—	—	35·7	—	—	—	—	—	—	—	—	—	59·5	76·6	78·1	98·2	− 1·4	—	—	—
38·0	38·0	34·3	—	—	—	—	—	—	68°·0	75°·3	36°·7	60·0	72·0	76·5	94·1	− 4·5	62·4	59·1	58
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
36·5	36·5	—	47·8	43·4	40·2	10·2	85°·7	83°·0	—	—	—	59·6	70·7	73·4	96·4	− 2·7	—	—	—
34·2	34·2	32·0	36·7	33·7	—	—	90°·5	85°·2	—	—	—	58·4	78·0	74·4	104·8	+ 3·6	—	49·6	49
37·4	37·4	33·0	—	44·2	38·7	11·0	92°·8	91°·0	67°·4	78°·8	33°·8	61·7	79·8	78·0	102·3	+ 1·8	62·7	58·0	56
38·0	36·9	32·8	48·5	46·0	35·0	6·8	90°·0	88°·2	65°·0	81°·9	33°·1	62·1	72·2	73·6	98·1	− 1·4	62·0	62·5	58
36·9	36·9	35·8	48·0	42·6	40·8	7·8	87°·1	85°·0	65°·6	81°·8	32°·6	55·7	69·4	72·8	95·3	− 3·4	60·9	59·7	57
38·4	38·1	33·8	45·2	42·1	35·0	5·2	89°·8	86°·9	64°·6	78°·0	37°·4	59·0	70·1	75·6	92·7	− 5·5	70·8	50·1	48
36·8	36·8	34·0	45·0	41·4	32·7	—	—	—	66°·3	80°·9	32°·8	58·8	72·4	76·4	94·7	− 4·0	65·2	58·8	57
36·9	36·9	35·9	—	43·3	—	—	—	—	—	—	—	58·4	77·5	—	—	—	77·8	48·5	46

100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH'}$ R	100 $\frac{O_2}{O_1}$, R	100 $\frac{O_2}{O_1}$, L	100 $\frac{O_2}{O_1''}$, R	100 $\frac{G_2}{G_1}$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{EH}{G_2}$	100 $\frac{fmb}{fml}$
—	43·1	40·4	82·9	82·9	89·9	—	34·9	30·0	—	—
—	—	—	—	82·1 ?	—	83·1	31·1	32·3	21·5	90·4
74·5	54·1	50·8	76·8	78·1	79·9	—	34·8	27·4	—	—
73·3	50·5	47·2	83·9	84·8	91·4	82·2	40·4	32·6	25·7	85·3
—	55·1	55·1	84·3	85·0	88·2	—	33·3	38·5	—	—
62·7	56·6	55·4	85·2	85·3	93·3	—	34·6	25·5	—	—
—	58·7	53·6	86·3	86·9	93·2	—	30·1	32·9	—	83·9
—	53·2	51·3	78·8	—	86·8	—	47·7	29·4	—	88·3
—	58·0	54·9	79·5	—	85·4	—	22·9	29·2	—	83·4
—	—	—	79·6	—	90·2	—	40·0	21·7	—	85·4
—	55·4	52·0	79·0	79·6	87·2	—	30·4	30·4	—	81·0
—	57·7	54·0	—	84·1	—	—	40·6	37·0	—	88·4
—	—	—	77·5	—	83·3	—	39·6	27·2	—	—
—	—	—	—	—	—	—	—	—	—	—
70·7	52·2	51·1	77·6	75·0	84·5	75·3	41·4	41·7	23·0	84·0
66·6	55·7	53·5	87·8	86·7	97·3	82·9	43·8	34·1	17·9	72·3
71·0	54·4	50·6	82·9	86·2	—	80·7	41·0	37·3	23·6	92·9
61·9	52·3	51·3	79·4	82·7	—	89·4	—	30·0	22·6	82·9
73·8	50·3	49·3	81·2	81·9	—	71·2	—	41·7	33·9	—
—	63·1	59·3	78·1	75·9	—	—	—	31·6	—	—
—	58·8	53·6	83·8	83·2	86·6	86·0	34·5	—	—	93·4
—	56·8	56·0	78·1	77·3	86·4	69·0	48·5	—	—	81·6
—	—	—	—	86·9	—	—	—	—	—	84·4
62·4	59·1	58·8	77·6	82·1	84·2	—	45·0	17·4	—	86·6
—	—	—	83·7	—	90·7	—	38·0	14·8	—	—
—	—	—	—	—	—	84·1	39·0	32·1	25·4	89·8
—	49·6	49·6	88·1	86·3	93·8	—	36·5	25·7	—	85·1
62·7	58·0	56·4	81·1	80·7	89·2	—	32·9	18·1	28·4	82·0
62·0	62·5	58·9	80·8	81·4	84·3	72·2	45·0	30·6	19·4	84·0
60·9	59·7	57·6	89·5	87·3	99·4	85·0	50·2	27·7	19·1	74·0
70·8	50·1	48·2	82·3	83·3	85·4	77·4	33·7	27·3	14·9	80·6
65·2	58·8	57·8	82·1	86·3	91·3	72·7	39·4	—	—	83·5
77·8	48·5	46·1	88·3	88·6	95·0	—	50·3	35·9	—	—

A STUDY OF THE NEGRO SKULL WITH SPECIAL REFERENCE TO THE CRANIA FROM KENYA COLONY.

By ELISABETH KITSON, B.A., Crewdson-Benington Student in Craniometry.

1. *Introduction.* I am indebted to Mr L. S. B. Leakey, Leader of the East African Archaeological Expedition, for the following description of how he collected the present series of Teita skulls in 1929 and for much valuable information about the customs of the tribe. He writes:

"In the course of my archaeological work in Kenya it soon became clear that we must obtain as many skulls as possible of the modern local natives in order to provide material for a comparative study between them and the Palaeolithic and Neolithic skulls which we were finding. Unfortunately, in the case of most tribes, this is an almost impossible task, but the presence of certain special customs among the Teita made a collection of skulls of this tribe possible.

The Teita, or Wa-Teita, are a small group of Bantu-speaking people living in the hilly country south of the Kenya-Uganda railway line in the neighbourhood of Voi*. They are essentially a hill tribe and until recently their principal centres were three hills called Dabida, Sagalla, and Kasegoa†. The last hill is no longer occupied. The Teita are allied to the Wa-Chagga, who live on the slopes of Kilimanjaro, and to the Wa-Pare, having certain customs in common with both these groups. But they have been studied very little from an anthropological point of view and for such information as I can give here upon various points directly connected with the skulls described in this paper I am indebted to the Venerable Archdeacon Maynard. He has been living and working among the Teita at Dabida for 28 years, and it was entirely due to him that I was able to collect the present series of skulls on the hills of that name.

The Teita practise a form of ancestor-worship, placing the skulls of the dead under the shelter of over-hanging rocks and in caves which thus become local shrines‡. Thither the people go to offer prayers and sacrifices to the spirits of their departed relatives. The normal procedure is to bury the dead, placing a stone immediately over the point where the head lies. After a period which may vary from 12 months to over two years, a small hole is dug and the skull—usually without the mandible—is withdrawn and taken to the family shrine. Adult males and females who have died in normal circumstances are treated in this way. If a person

* See map (Fig. 1) below.

† No two of these three hills are more than about 20 miles apart

‡ Plate IV *d*, supplied by Mr Leakey, shows a number of skulls arranged in front of one of these shrines.

meets with a violent death away from home, it is considered enough if a limb-bone can be recovered and placed in the shrine; failing this, a sheep's skull may in extreme cases be substituted.

To-day many of the Teita have become Christians, especially round the Church Missionary Society station where Archdeacon Maynard has been working, hence they no longer pray and sacrifice to the skulls of their ancestors. Through the kindness of Archdeacon Maynard, Chief Thomaso was approached and he supplied a guide who went with me to the various shrines in his district (all within a radius of about half-a-mile from his village) to show me which skulls I might take away. These were either the skulls of people whose relations are now all Christians, or skulls whose owners had died so long ago that no one could remember them as living people. In this way I was able to collect some 120 skulls of both sexes from various shrines. All the shrines were within the area of the same clan, so, the Teita being patrilineal and patrilocal, the male skulls almost certainly belong to the same clan with the possible exception of a few which represent prisoners or slaves from other tribes. Most of the female skulls probably represent women of other clans or sub-clans living in the Dabida Hills, for on the whole the Teita are exogamous, though endogamy is not entirely unknown, cases having come to Archdeacon Maynard's notice. It is therefore possible that some of the female skulls belong to the same clan as the males. A few others may have been women slaves captured from other tribes and subsequently married to Teita men. In this connection we may note that the Teita do not practise extraction of the incisors, a custom common in many other East African tribes and which takes place as soon as the permanent incisors are in position. Among the skulls which I brought home a few clearly show artificial extraction of the incisors at an early age, and it is therefore reasonable to regard these as representing slaves from other tribes and not true Teita. It is impossible, however, to say which tribe they came from.

Like other East African agricultural tribes, the Teita grind their maize and other corn into flour between two stones. The local stone being granite or schist, their food consequently contains a considerable quantity of hard grit. This may well account for the bad state of the teeth which are much worn in the majority of the skulls.

In some of the skulls there is a very pronounced flattening of the nasal bones in the region of the nasion. This I am inclined to regard as being due to accentuation of the naturally flat nasal bones of the African negro by unintentional artificial deformation. Among most East African tribes the mother 'blows' her child's nose by placing her thumb near the nasion, pressing it firmly down and drawing it towards the child's mouth thus expressing the mucous. As this is often done from a day or two after birth it may well be the cause of excessive flattening of the nasal bridge. But Archdeacon Maynard assures me that this is not done by the Teita, so I am inclined to believe that the skulls which show this peculiar feature may also have come from other tribes, though this is not so certain as in the case of those skulls which have the incisors extracted.

Many of the skulls are deeply marked, especially on the face, by the gnawings of rodents. The large bush rats and porcupines are very probably responsible for this.

The total number of the Teita is at present only about 40,000 and, with the exception of the few skulls which probably represent slaves, the present series may be regarded as typical and fully representative of the tribe."

Mr Leakey also informed me that none of the skulls he collected is more than three or four generations old, the oldest coming from the "E" shrine.

The total number of skulls is 123 adults and 1 child, each skull being marked with a letter (A, B, C etc.) to denote the shrine from which it came. Twelve different shrines are represented and the following are the numbers of skulls obtained from each: A: 11, B: 3, C: 2, D: 3, E: 67, F: 9, G: 10, H: 1, J: 6, K: 4, L: 7, M: 1. The shrine number and serial number of each skull are given in the tables of individual measurements at the end of this paper.

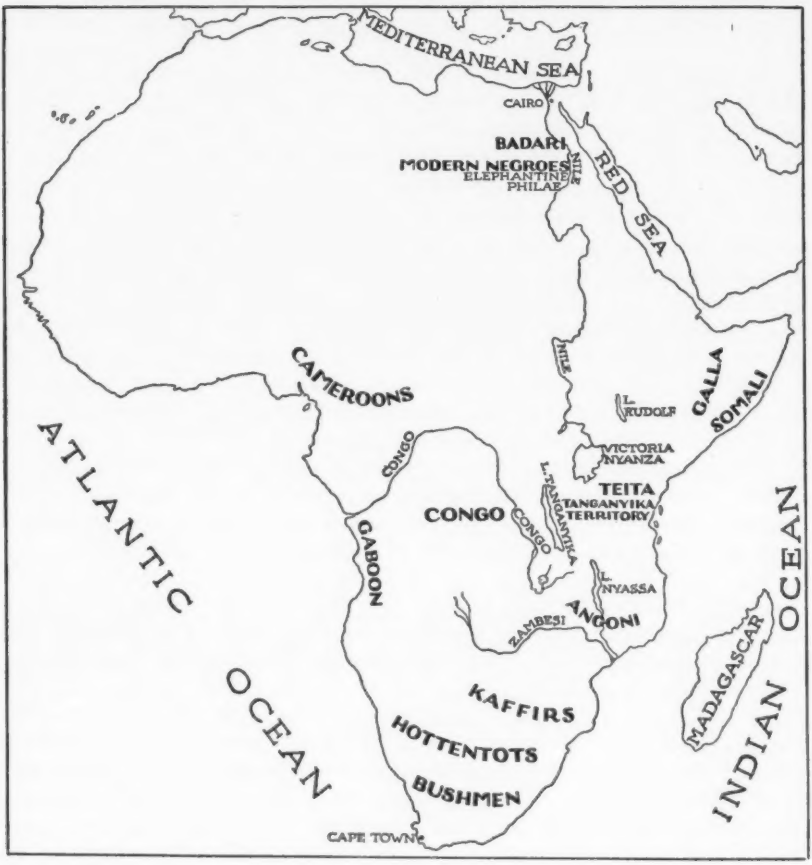
I have also measured a short series of 15 skulls from a cave near the Naishi River, Kenya Colony. Two of these were presented to the Biometric Laboratory by Mr Leakey and the rest were kindly lent by the authorities of the British Museum (Natural History). I have compared this series with the Teita as it was thought that there might be some connection between the two groups. The Teita are believed to have reached their present locality from further north, and the fact that the skulls were deposited in a cave and that there is a place called Elmenteita in the district seemed suggestive. But it will be shown later that there appears to be no close resemblance between the two groups. Mr Leakey writes the following account of the discovery of the Naishi River skulls: "On February 7th, 1928, Dr A. S. Parkinson, of the British Museum East African Expedition, was conducted by Mr A. M. Cooper, of the Kenya Forestry Department, to a rock shelter on the west bank of the Naishi stream near the summit of the Mau hills*. Mr Cooper had already found two human skulls here and Dr Parkinson soon discovered some more. Most of the skulls found were in a condition suggesting that they were possibly two or three hundred years old, though one child's skull and some ox bones found in the same shelter appeared to be very recent. Dr Parkinson suggested that the shelter had been used as a place for depositing skulls for some hundreds of years. He also says†: 'It seems evident that two entirely distinct races have been placed in these rock shelters....We found four stone bowls in close association with the bones, and within a few feet an ellipsoidal pebble of coarse quartzite...showing marks of abrasion at either end. Doubtless it was used as a pestle.' During the 1928—29 season I twice visited this site, Mr Cooper very kindly providing a guide on the first occasion. Some more human remains were recovered from the surface dust of the shelter, but detailed excavations had to be postponed."

* The position of the site may be indicated on the map (Fig. 1) as nearly under the left foot of the A in Teita.

† "Eighth Interim Report of the British Museum East African Expedition," by J. Parkinson, Feb. 13th, 1928. (MS.)

Until further evidence is available, it will be impossible to date these skulls, or to assign them to a particular tribe. The measurements of the few specimens at our disposal have been given in this paper in the hope that more may be added to them in the near future.

FIG. 1. MAP SHOWING DISTRICTS FROM WHICH THE CRANIAL SERIES DEALT WITH IN THIS PAPER WERE OBTAINED.



Our main purpose has been to give a detailed description, from the biometric standpoint, of the Teita and Naishi River skulls and to compare the former, which make one of the longest negro series at present in Europe, with all the other sufficiently long negro and closely related series for which measurements have been published.

2. *The Nature of the Series from Kenya Colony.* The Kenya (Teita) series is well preserved on the whole though very few of the skulls are actually complete. The condition of the bone is in most cases fairly good, but perhaps not as good as it might have been considering that the skulls are very probably not more than a hundred years old. This is largely due to the native method of hoarding skulls in caves without proper burial, thus exposing them to various destructive agencies. Many of them had been worn by dripping water, or had holes eaten in them by roots, some of which can be seen still clinging to the bone. A number of specimens bear teeth marks, usually on the face, the result of gnawing by porcupines or large rats. This condition was observed by Crewdson-Benington in the series from the Congo and Fernand Vaz*. All the specimens are discoloured.

The sexing was done by Professor Pearson and Dr Morant. They divided the series into 57 adult males and 66 adult females, each group including doubtful specimens. Approximately equal numbers of males and females were to be expected. Several writers have remarked on the difficulty of sexing negro series, but this was by no means excessive in the case of the present one. There is a considerable number of large muscular males and as many small and feeble females.

In Table I the sex ratios of the Teita are compared with those of one negro, one Egyptian, and two European series, all means used being based on more than forty skulls. It is expected that the ratios will be greater for the more advanced races, but these results show very little difference between the African and European series chosen. The Teita ratios come nearest to the Egyptian *E*, being almost identical for all characters except *B'*.

TABLE I.

Sex Ratios (♂ Mean/♀ Mean) for the Teita and other Series (all Means used based on more than 40 skulls).

Characters	Teita	Gaboon† (1864)	Egyptian *E*‡ (26th—30th Dynasties)	English§ (Farringdon Street)	Modern German‖ (Altbayerisch)
L	1·046	1·046	1·047	1·040	1·041
B	1·024	1·038	1·025	1·049	1·045
LB	1·051	1·046	1·053	1·045	1·047
B'	1·039	1·039	1·027	1·038	1·049
H' or *H*	1·038	1·041	(*H*) 1·038	1·059	(*H*) 1·045
S	1·037	1·040	1·034	1·046	1·033
U	1·037	1·042	1·038	1·044	1·046

* R. Crewdson-Benington: *Biometrika*, Vol. VIII. p. 325 and Plate XXVI.

† R. Crewdson-Benington: *Biometrika*, Vol. VIII. pp. 298—300.

‡ Karl Pearson and A. G. Davin: *Ibid.*, Vol. XVI. p. 337.

§ B. G. E. Hooke: *Ibid.*, Vol. XVIII. pp. 28, 29.

‖ J. Ranke: *Beiträge zur Anthropologie und Urgeschichte Bayerns*, Bd. III. pp. 194—197; Bd. V. pp. 74—77 and pp. 88—91.

Comparisons are made in Table II between the variabilities of the Teita and of the long Egyptian *E* series of 26th—30th Dynasty skulls. All the constants given there for the negroes are based on more than 40 specimens. Comparing coefficients of variation of the absolute measurements and standard deviations of the indices, we find for the males that the Teita variabilities are the greater for 10 characters and the lesser for 12. Hardly any of the differences are significant, however, the only ones exceeding 2·5 times their probable errors being in the case of *Oc. I.* (Δ/P.E.Δ = 7·59), *fmb* (Δ/P.E.Δ = 3·33) and S_2 (Δ/P.E.Δ = 2·55). The Egyptian variability is the

TABLE II.

Variabilities of the Teita and Egyptian E Series.*

Characters	Teita				Egyptian *E* (26th—30th Dynasties)	
	Standard Deviations		Coefficients of Variation		Coefficients of Variation	
	♂	♀	♂	♀	♂	♀
L	6·07 ± ·40	4·98 ± ·31	3·30 ± ·22	2·84 ± ·18	3·09 ± ·05	2·66 ± ·05
B	4·33 ± ·28	4·39 ± ·27	3·34 ± ·22	3·47 ± ·21	3·43 ± ·05	3·34 ± ·07
B'	4·50 ± ·29	3·94 ± ·24	4·69 ± ·30	4·26 ± ·26	4·28 ± ·07	4·11 ± ·08
H'	4·47 ± ·32	4·04 ± ·28	3·42 ± ·25	3·21 ± ·22	—	—
H	4·18 ± ·32	3·88 ± ·27	3·21 ± ·24	3·11 ± ·22	3·75 ± ·06	3·39 ± ·07
OH	3·91 ± ·27	3·41 ± ·23	3·50 ± ·24	3·18 ± ·21	3·63 ± ·06	3·32 ± ·07
LB	4·24 ± ·21	3·45 ± ·24	4·17 ± ·31	3·57 ± ·25	3·90 ± ·06	3·65 ± ·07
S_1	5·30 ± ·35	5·85 ± ·36	4·17 ± ·28	4·76 ± ·30	4·88 ± ·08	4·56 ± ·09
S_2	8·79 ± ·57	6·42 ± ·39	6·94 ± ·45	5·26 ± ·32	5·77 ± ·09	5·19 ± ·10
S_3	7·88 ± ·57	5·63 ± ·38	6·94 ± ·50	5·20 ± ·35	5·91 ± ·10	6·06 ± ·12
S	13·85 ± 1·02	11·66 ± ·80	3·78 ± ·28	3·30 ± ·23	3·36 ± ·05	2·92 ± ·06
U	13·51 ± ·88	12·09 ± ·75	2·66 ± ·17	2·47 ± ·15	2·65 ± ·04	2·35 ± ·05
Q	8·56 ± ·59	9·21 ± ·62	2·84 ± ·20	3·18 ± ·21	3·22 ± ·05	2·98 ± ·06
fml	2·47 ± ·18	2·37 ± ·17	6·77 ± ·50	6·65 ± ·47	6·95 ± ·11	6·37 ± ·12
fmb	1·68 ± ·13	1·93 ± ·13	5·70 ± ·43	6·73 ± ·46	7·18 ± ·11	6·96 ± ·14
NH, L	3·08 ± ·21	2·96 ± ·20	6·23 ± ·42	6·28 ± ·43	5·65 ± ·09	5·31 ± ·10
NB	1·79 ± ·12	1·94 ± ·13	6·49 ± ·45	7·17 ± ·50	7·27 ± ·12	6·98 ± ·14
O_1L	1·64 ± ·11	1·55 ± ·11	3·74 ± ·26	3·70 ± ·25	4·06 ± ·07	3·97 ± ·08
O_2L	1·97 ± ·14	1·67 ± ·11	5·80 ± ·40	5·01 ± ·34	5·56 ± ·09	5·62 ± ·11
					Standard Deviations	
100 *B*/*L*	3·01 ± ·20	2·47 ± ·16	—	—	2·68 ± ·06	2·56 ± ·05
100 *H'*/*L*	2·53 ± ·18	2·37 ± ·16	—	—	—	—
100 *B*/*H'*	4·50 ± ·32	4·20 ± ·29	—	—	—	—
Oc. I.	2·10 ± ·15	2·52 ± ·17	—	—	3·30 ± ·05	3·20 ± ·06
100 *NB*/*NH, L*	3·74 ± ·26	4·73 ± ·34	—	—	3·82 ± ·06	3·77 ± ·07
100 O_2/O_1L	4·39 ± ·31	3·98 ± ·27	—	—	4·95 ± ·08	4·67 ± ·09

* All the constants for the Teita series in this table are based on 40 or more skulls and the numbers available for each character can be ascertained from Tables IV and V. All the Egyptian *E* constants are based on more than 500 skulls.

greater in the first two of these cases. For the females the Teita variabilities are the greater for 12 characters and the lesser for 10, the only differences of any consequence being for *Oc. I.* (Δ/P.E.$\Delta = 3{\cdot}77$)—the Egyptian constant being the greater—and $100NB/NH$, L (Δ/P.E.$\Delta = 2{\cdot}77$), the Teita constant being the greater. Judging from all the characters, we may conclude that the two series are almost equally variable, while the negroes have a peculiarly constant occipital index. It is known that this Egyptian series is decidedly more homogeneous than almost all European ones available. A comparison similar to the above was made between the Teita and Farringdon Street constants for the 25 characters given in Table II and this shows that the negro variabilities only exceed the English for five characters (S_2, S_3, S, NH, L, and O_1L) in the case of the males and three (NH, L, *Oc. I.* and $100NB/NH, L$) in the case of the females. There is every reason to believe that the Teita series is racially homogeneous, but in view of the fact that a single tribe is represented, its variability is by no means outstandingly small. It will be shown later (see Table VI and p. 295 below) that it is of the same order as the variabilities of several other negro series, of which some were made up by pooling a number of different tribes. Judging from the appearance of the skulls alone, the Teita series is racially as pure as any with which the craniometrician usually deals, and there can be no justification for sub-dividing it.

A comparison of the variabilities of the male and female samples distinguished is of particular interest. It has been explained in the introduction that the Teita tribe is patrilocal and that the male skulls of this series almost certainly belong to the same clan, with the possible exception of a few which may represent prisoners or slaves from other tribes. Apart from similar exceptions, the females would represent several clans or sub-clans living in the same hills and a few may belong to the same clan as the males. Given these conditions operating for any length of time within a very limited population, it is not to be expected that the variability of the female series will be greater than that of the male. A comparison of the variabilities in Table II shows that there is only a small sexual difference in this respect. Using coefficients of variation of the absolute measurements and standard deviations of the indices, there are found to be 17 characters having the male variability the greater, and eight the female. The only differences likely to be significant are in the case of S_2(Δ/P.E.$\Delta = 3{\cdot}04$) and S_3(Δ/P.E.Δ 2·85), the male constant being the greater in both cases. So we may conclude that the female group is the *less* variable. This slightly greater relative variability of the male population has been observed for other series. It is rather more accentuated in the case of the Egyptian *E* series, there being twenty male constants greater and only two less for the characters in Table II. Thus the marriage customs of the Teita do not appear to have made the relative variabilities of the sexes within a single clan any different from those commonly found in other populations.

All the skulls (except one—B 3, a child of about 6 years) are adult with the third molars erupted and the basal suture closed. There are no definitely senile skulls. The approximate age groups in the following table are formed from the condition

of the three principal sutures—the coronal, sagittal, and lambdoid—in each skull. It is a known fact that the sutures close at an earlier age in male than in female skulls, so there is no evidence of a sexual difference between the distribution of ages at death.

Sex	All sutures open	Sutures beginning to close or partly closed	All sutures closed or nearly closed	Total No.
♂	12 (24·5 %)	21 (42·9 %)	16 (32·7 %)	49
♀	38 (62·3 %)	19 (31·1 %)	4 (6·6 %)	61

Of the 15 skulls from the Naishi River cave six are nearly complete and the others are fragmentary. The bone is well preserved and of a light reddish-brown colour in the majority of cases. All but one of the specimens are supposed male, most of them being of a large muscular type. Probably a selection was made of the better preserved skulls. No standard deviations were calculated as the series is too short, but judged by their general appearance the skulls form a racially homogeneous group. The female specimen has the coronal and sagittal sutures nearly obliterated, but the lambdoid clearly open. One of the males (No. 8) is a young adult with the third molars not fully erupted; one other has the three principal calvarial sutures open, six have the sutures partly closed and five have them all obliterated, closed or nearly closed.

The measurements of the Teita series show that all the more salient characteristics of a negro type are present, such as a high nasal index, large nasal and small profile angles (indicating marked, though not exaggerated, prognathism), a low upper facial ($100G'H/GB$) and low simotic and dacryal indices. All these are characters of the facial skeleton. None of the means is extreme except the low simotic index ($100SS/SC$) for which there is little comparative material. It has been suggested in the Introduction that the flatness of the nasal bridge in this series may have been exaggerated by artificial deformation. This does not seem to be probable, however, as the standard deviations of the character are low: the male value is $7{\cdot}76 \pm {\cdot}59$ and the female $7{\cdot}07 \pm {\cdot}48$, while for the Farringdon Street English series the corresponding standard deviations are $12{\cdot}75 \pm {\cdot}68$ and $10{\cdot}19 \pm {\cdot}56$. It can be seen from the sagittal type contours (Figs. 7 and 8) that there is only slight sub-nasal prognathism.

It has been suggested that a high occipital index is a characteristic of the negro skull*, but the best negro data available at that time related only to West African races. The Teita means are as low as for many Egyptian, European and Asiatic series and others can now be given for East African negroes which are of the same order (see Table VI below).

The size of the Teita skull is on the whole very small as is shown by the capacity and the three major calvarial arcs. No one of these is smaller, however,

* See *Biometrika*, Vol. xvi. pp. 332—335.

than the means found for some other negro series. When compared with other races from all parts of the world*, the calvarial breadth B seems to be the outstandingly small character. Only two lower male means for B could be found—for Veddahs and for Northern Territory Australians. This makes the cephalic index very low, but still lower means can be found for some races of Oceania. The index $100\,B/H'$ is also low, but not extreme, and $100\,J/B$ is another which is influenced by the extremely small size of B. The mean of the last index for males is 101·0 (27) and for females 97·4 (35), the sexual difference being that normally found for other series. No other races in Africa, and none in Europe or Asia, are known to be phaenozygous, i.e. to have this index greater than 100. The condition is found in a few American races (Greenland Eskimos (103·8) and modern Patagonians (102·1)) and it is more frequently met with among the races of Oceania (Loyalty Islanders (104·9), Northern Territory Australians (104·8), New Caledonians (103·3), and New British (101·1)). It appears to be a primitive characteristic, though like all such, it is not capable of distinguishing all primitive from all advanced races. The Teita orbital and palatal indices are low, but not extremely so.

Male and female means for our longer series are given in Tables IV and V (pp. 292—294). There is a satisfactory agreement between the indices and angles for the two sexes. Significant differences are normally found between the male and female means of some indices, when these are based on considerable numbers, and it is customary to find that the female cephalic, occipital and orbital indices exceed the male, as in the present case. Differences between the nasal and palatal indices are also significant and this is unusual. For $100\,NB/NH'$ the female mean exceeds the male by 3·5 and this is 4·4 times the probable error of the difference; for $100\,G_2/G_1$ the female mean exceeds the male by 4·1 and this is 3·9 times the probable error of the difference. Judging from the comparison of all the indices and angles together, however, the agreement between the sexes is close and we shall be justified in considering that the two series represent the same homogeneous racial type.

It was thought advisable to compute the means for two smaller groups of the Teita series and these are given in Table III. The skulls from the E cave are known to be somewhat older than any of the others. The means for this group and for the total series are seen to be very close indeed, however, and it is probable that none of the differences is significant. It is suggested in the Introduction that the specimens which have had one or both central incisors knocked out are not true Teita, as the custom is not practised by that tribe. There are few skulls exhibiting this condition and, as the numbers are so small, it is probable, again, that the means for them do not differ significantly from the corresponding ones for the total series. The evidence suggests that the small group represents closely related types if not that of the Teita proper, and there can be little harm in pooling all the material.

* The comparative material used here is that for which references are given in *Annals of Eugenics*, Vol. II. 1927, pp. 378—381, with the addition of a few other series.

TABLE III.

Mean Measurements of different Groups of Kenya Colony Skulls.

Characters	Teita: whole series		Teita: *E* cave		Teita: skulls with central incisors knocked out		Naishi River
	♂	♀	♂	♀	♂	♀	♂
L	183·9 (53)	175·7 (58)	184·1 (27)	175·5 (29)	180·4 (5)	174·3 (9)	189·9 (9)
B	129·6 (54)	126·6 (61)	128·9 (28)	126·7 (33)	128·0 (5)	125·0 (9)	138·3 (8)
H'	130·8 (44)	126·0 (49)	131·1 (21)	125·5 (23)	130·1 (5)	124·8 (8)	129·3 (6)
J	130·9 (27)	122·9 (35)	131·7 (11)	123·4 (18)	127·0 (3)	121·9 (6)	—
100 *B*/*L*	70·6 (53)	72·0 (56)	70·3 (27)	72·2 (29)	71·0 (5)	71·7 (9)	73·6 (6)
100 *H'*/*L*	71·4 (43)	71·6 (48)	71·4 (20)	71·2 (22)	72·2 (5)	71·8 (8)	68·7 (5)
100 *B*/*H'*	98·9 (44)	100·3 (49)	98·0 (21)	101·0 (23)	98·4 (5)	99·9 (8)	107·4 (5)
100 *NB*/*NH'*	58·3 (36)	61·8 (41)	57·4 (17)	60·6 (19)	60·8 (3)	63·9 (8)	53·0 (5)
100 O_2/O_1, *R*	77·8 (40)	79·3 (48)	76·6 (22)	79·5 (26)	79·3 (4)	79·3 (7)	81·6 (6)
Oc. I.	59·5 (44)	60·7 (51)	59·5 (22)	60·8 (26)	60·5 (5)	61·1 (9)	59·1 (11)

The figures in brackets give the number of crania measured.

Means of the short male series from the Naishi River cave are also given in Table III. These skulls are remarkably similar to one another and they differ clearly from the Teita specimens in being larger and in having more retreating frontal bones. The most marked difference is found between the calvarial breadths (*B*): the Teita male mean is 129·6 (54) and the range is 119·5—139, while the mean for the Naishi River series is 138·3 (8) and the range is restricted between 135 and 141·5. There is thus little over-lapping of the distributions and, in spite of the small numbers, the difference between the means is markedly significant. Without further material it is impossible to determine the relationships of the shorter series, though it appears to represent a peculiar type. The occipital index is low as for the other available East African negro series.

3. *Remarks on Anomalies.* Owing to the incomplete nature of several of the skulls, it was necessary to determine separately, in the case of each anomaly, the total number which can now be examined for the condition.

(*a*) *Sutures.* Remarks on the condition of the coronal, sagittal and lambdoid sutures for each skull are appended to the tables of individual measurements. If present, any one, or all, of these sutures are open, unless otherwise stated. A rough appreciation of the age constitution of the samples deduced from this evidence has been considered above. The following summary for the Teita series relates to the order in which the principal sutures are closing:

Sex	Sagittal closing first	Coronal closing first	Sagittal and coronal closing together before lambdoid	Sagittal and lambdoid closing together before coronal	Total
♂	11	12	13	1	37
♀	2	13	4	0	19

Both the sagittal and coronal sutures show a definite tendency to close well before the lambdoid, and there is not a single case of the last closing before the other two. The coronal shows a slight tendency to close before the sagittal suture in the males, and this is definitely so for the females. For European races the normal order of closure appears to be: sagittal—lambdoid—coronal. There are certainly racial differences in this respect, as Gratiolet noted, but adequate statistics are still wanting.

The complexity of the sutures varies very considerably. In a number of cases they are extremely simple, whilst in others they are comparatively complex, or, in a few cases, very complex. Though the standard of complexity is probably rather lower than for European skulls, it cannot be said that the evidence of this series lends strong support to the theory that sutures are much simpler in primitive than in advanced races.

Only one case of any unusual condition in the closing of the sutures was noted. In (J 5 ♀)* the coronal suture was closed on the right side but clearly open on the left.

Out of 122 possible adult Teita skulls only one example of metopism was observed. In this (E 42 ♂) the metopic suture is complete, but beginning to close. It meets the coronal to the left of the bregma, thus causing articulation of the right frontal with the left parietal. In a few other cases a trace of the suture can be seen near the nasion, but this is never more than about 10 mm. long. The percentage of metopic skulls is thus very low as for other primitive races†.

There are three cases of fronto-temporal articulation (E 20 (L.), F 7 (R. and L.) and F 9 (R. and L.)) out of 36 possible male skulls, and two (E 24 (L.) and E 35 (R. and L.)) out of 41 possible female skulls, giving a percentage of 6·5 for the series. This is not as high as is usually expected for negro races‡.

One skull (F 6 ♂) shows traces of a suture between the ex- and supra-occipitals on both sides.

A horizontal suture across the malar bone occurs in four skulls—E 18 ♂ (traces), G 3 ♂, F 1 ♀ (left side only), and K 4 ♀ (parts). It is more frequent than usual in this series. There are 43 male skulls with both malar bones intact (4·7 °/₀ affected) and 41 female (4·9 °/₀ affected).

Of the 12 male skulls from the Naishi River cave with frontal bones intact not one is metopic. The female shows a clear trace of the suture above the nasion, but this is only 30 mm. long. Out of eight male skulls, the sagittal is closing before the other two calvarial sutures in five cases, the coronal is first in one case, and in two the sagittal and coronal are closing together before the lambdoid. The lambdoid is generally the last to close, as in the Teita series. There is one case of

* The cave numbers of individual skulls are given in this section and the corresponding serial numbers are given in the tables of individual measurements. It should be remembered that all individuals from the same cave are probably closely related to one another.

† Cf. Le Double: *Traité des Variations des Os du Crâne de l'Homme*, p. 151.

‡ Cf. *Ibid.*, p. 303.

fronto-temporal articulation (No. 7, R. side only) among five male skulls. This anomaly does not occur in the female. The sutures are moderately complex in general.

(*b*) *Supernumerary Bones.* Three definite cases of interparietal bones were found out of 51 possible cases of male Teita skulls and one out of 59 possible cases of female skulls, a percentage of 3·6 for the whole series, but all affected specimens come from the E cave. In E 41 ♂ and E 51 ♂ only the *os pentagonale* is present, while in E 20 ♂ the *os triangulare* is visible on the right side, though the suture dividing it from the *os pentagonale* is obliterated. In E 4 ♀ the *os pentagonale* is separated by a distinct suture from the *os triangulare* on the right. In no case is the *os triangulare* present on the left side.

Wormian bones (excepting ossicles of lambda) occur in the lambdoid suture in 28 out of 44 possible male skulls and in 32 out of 59 possible female skulls, a percentage of 58·3 for the series. Only one skull (K 3 ♂) has an ossicle in the sagittal suture, this being situated immediately above the lambda. No ossicles, except for a few extremely small ones, were found in the coronal suture, and there are no ossicles of the bregma. Ossicles of the lambda occur in two out of 45 possible male and in 14 out of 58 possible female skulls. There appears to be a definite sexual difference in these numbers*, the ossicles being far more numerous, as well as more complex, among the females than among the males. In some of the females they are divided into two or four parts, and in one case (E 59 ♀) into six, while in both the male cases they are small and undivided.

Epipteric bones are present in eight out of 36 possible male and eight out of 41 possible female cases. There are 11 altogether (7 R. and 4 L.) in the male group and 12 (5 R. and 7 L.) in the female.

In the series from the Naishi River cave none of the 12 male skulls shows any trace of interparietal bones. In the female specimen the *os pentagonale* is separate, being nearly square in form and with complex sutures. Among the male skulls in which the sutures at the pterion can be seen, two have epipteric bones on the right side out of seven possible, and four have epipteric bones on the left side out of nine possible. A few Wormian bones occur in the lambdoid sutures, but none in the coronal or sagittal. There are no ossicles of bregma.

(*c*) *Teeth.* The teeth, on the whole, are in a remarkably bad condition, and many appear to have been lost before death even in quite young individuals. There are 37 skulls having one or more carious teeth *in situ*. The majority of the teeth are considerably worn and many are markedly so. One male (J 3) and seven female (A 1, C 1, E 27, E 54, E 67, J 1 and J 4) fully adult skulls appear to have had no upper third molars erupted on either side. In one other female (E 8) the upper third molar is present on the left side, but absent on the right. A small and definitely anomalous third molar occurs on the left side in E 59 ♀. No other dental anomalies were observed. A male specimen (E 39) has the central incisor

* Cf. Le Double, *op. cit.*, p. 63.

on the left side with a curved lower edge. This was probably produced artificially by filing, but there are no other such cases in the series. In several skulls either one, or both, of the central incisors have been knocked out, leaving when healed a sharp edge to the alveolar border which can usually, though not always, be distinguished with certainty from the closed socket of a tooth lost naturally. Out of 47 possible male skulls, five (A 2 (L.), F 4 (R. and L.), G 1 (R.), G 2 (L.) and J 3 (R. and L.)) have lost one, or both, central incisors, almost certainly by artificial means. Nine cases occur (A 1 (L.), A 3 (R.), E 16 (R.), E 26 (R.), E 27 (R. and L.), E 37 (L.), F 1 (L.), F 2 (R.) and J 5 (L.)) among 46 possible female skulls. The custom of knocking out the central incisors thus occurs with very moderate frequency in this series, the percentage being 15·1. Neither does it appear to be more common in skulls from any particular cave, and, though the loss of one tooth is more common than the loss of both, the condition is found on both sides with approximately equal frequency.

In the Naishi River skulls the teeth seem less worn, or decayed, than in the Teita series. Only one diseased tooth—a molar—was found among those *in situ*. In two cases the central incisors had apparently been knocked out intentionally, while in three others this had not been done.

(*d*) *Base of Skull.* The relative sizes of the jugular foramina for the Teita skulls are compared in the following table:

Sex	*JR*	*J=*	*JL*
♂	27	8	10
♀	28	9	13

From this it can be seen that the foramen is usually larger on the right side, a similar result having been found in all series previously examined. Very few anomalies of the base of the skull were observed. There is not a single case of isolated pre-condyles, the nearest approach to the condition being in F 7♂. There are no par-occipital processes. A pharyngeal fossa occurs in two male (E 7 and L 2) and two female (E 22 and E 49) skulls, all these being small and shallow depressions. Other minor anomalies noted are a low ridge connecting the anterior angles of the foraminal condyles (E 5♂), a slight basi-occipital incisure on the right side in E 20♂ and on the left side in J 4♀.

In the Naishi River series the distribution of the relative sizes of the jugular foramina is as follows:

Sex	*JR*	*J=*	*JL*
♂	6	1	1
♀	1	—	—

The only basal anomaly noted is in No. 7♂ where the anterior border of the foramen magnum is formed by two small but distinct surfaces which appear to have been articular.

(*e*) *Other Anomalies.* In noting cases of tympanic perforation among the Teita skulls any hole through which a wire 0·28 mm. in diameter could be passed was counted. The perforations are numerous and they vary considerably in size. Out of 52 possible male cases 24 are affected, 29 perforations occurring on the right and 19 on the left side; while out of 60 possible female cases 32 are perforated, 39 times on the right and 33 times on the left side. Thus, taking the series as a whole, 50 % of the skulls show some degree of tympanic perforation. It is not by any means confined to the old skulls in this series, though some anatomists have asserted that it normally occurs as a result of the thinning of the plate in old age*. The following table shows the percentage of perforated skulls occurring in each age group, the totals available for these groups having been given on p. 278 above:

Sex	Sutures open	Sutures beginning to close or partly closed	Sutures closed or nearly closed
♂	50·0 %	52·4 %	43·8 %
♀	52·6 %	42·1 %	100·0 %

From this it appears that for both sexes about half the skulls in the two younger groups are affected, while not quite half are affected in the males of the oldest group. There are only four females in the oldest group, all of these being affected, so that the large percentage here may be misleading. The number of skulls is too small for any generalisation regarding the relation of age and perforation, but it seems evident that in this series the younger skulls are as likely to have tympanic perforation as the older ones. The perforations appear to be as large, too, for the young as for the old specimens.

Only one exostosis was found (K 4 ♀). This is at the obelion, being approximately circular in form with a diameter of 16 mm. and a maximum height of about 1 mm. It is divided in half by the sagittal suture. A few small and superficial healed wounds occur, the most extensive being on the left side of the frontal bone in J 6♂. There are no cases of marked or even moderate asymmetry, and none of palatine torus.

The nasal bones are extremely variable in form. There are 46 male and 46 female skulls with more or less complete nasal bones and among them clear departures from the normal arrangement were noted in the following cases:

(1) Suture between nasal bones meeting naso-maxillary instead of naso-frontal suture—F 3 (♂).

(2) Suture between nasal bones meeting fronto-nasal suture two or more mm. to the right or left of the nasion—F 9 (♂), G 1 (♂), K 1 (♂), D 1 (♀), and H 1 (♀).

* Cf. Le Double, *op. cit.*, p. 324.

(3) Absence of suture between nasal bones with naso-maxillary and calvarial sutures open—E 26 (♀), F 1 (♀), L 6 (♀).

(4) Absence of nasal bones with calvarial sutures open—L 5 (♀) (Plate IV A) has an open and simple suture dividing the maxillary bones; E 55 (♂) (Plate IV B) and E 58 (♂) (Plate IV C) have roughened surfaces and what appear to be wormian bones between the maxillaries.

(5) Naso-maxillary sutures completely obliterated, except for traces of their upper parts (the fronto-nasal suture and suture between the nasal bones being clearly open), and traces of furrows starting from near the lower external angles of the nasal bones and converging until they nearly meet in the median sagittal plane —G 7 (♂).

(6) Healed fracture of the lower halves of the nasal bones—E 4 (♀).

The unusual number of nasal anomalies among these Teita skulls may possibly be a tribal, rather than a racial characteristic, as we are here dealing with a few closely interrelated groups. No anomalous nasal bones were found in the Naishi River series.

(*f*) *Disease.* A pathological condition which is unusually common in the Teita series may also be a tribal, rather than a racial characteristic. This is a roughening and irregularity of the glenoid surface supposed due to arthritis. Among the males, the following skulls show this condition: E 5 (R. acute: Plate V B), E 19 (L. slight), E 20 (R. acute: Plate V C, L. marked), E 30 (R. and L. slight), E 39 (R. and L. slight), E 47 (L. marked, R. missing), E 62 (R. and L. slight), G 3 (R. and L. acute), G 9 *b* (R. and L. slight), J 3 (L. slight). The female specimens affected are: A 11 (R. and L. marked), E 21 (R. slight), and E 50 (L. marked). The males appear to have been more affected than the females.

The only other sign of disease noted, other than dental caries, is in the case of a male specimen (E 19: Plate V A). The parietal bones are pitted with small depressions over an area which is roughly triangular in form, having one angle at the bregma and the other two near the asteria. The largest depressions are round the vertex and all posterior to the obelion are exceedingly small. This condition is supposed due to periostitis.

4. *Comparative Material.* An attempt was made to collect all the measurements of African negro and related series for 25 or more male skulls which are available, and for which adequate measurements have been published. It is an advantage when all the skulls in the series have been described by the same worker, but, in order to make the totals sufficiently large, it was necessary in several cases to pool the data provided by a number of craniometricians. The following abbreviations are used in the list of sources below: *A. f. A.* = *Archiv für Anthropologie*, *A. p. A.* = *Archivio per l' Antropologia e l' Etnologia*, *A. S. D.* = *Die anthropologischen Sammlungen Deutschlands*, being supplements to *A. f. A.*, *Bm.* = *Biometrika*, *B. (M.) S. A. P.* = *Bulletins (et Mémoires) de la Société d'Anthropologie de Paris*, *J. (R.) A. I.* = *Journal of the (Royal) Anthropological Institute.*

(1) *Angoni.* F. Shrubsall: "A Study of A-Bantu Skulls and Crania." *J. A. I.*, Vol. XXVIII. 1898, pp. 55—94. A series of 25 male skulls, now in the British Museum (Natural History), obtained by Sir H. H. Johnston from M'ponda's Town at the south end of Lake Nyassa. The donor describes them as being "the heads of a slightly mixed negro race mainly belonging to the Anyanja stock with a slight Zulu intermixture. It is possible, however, that one or more of them may be Yaos mixed with Arab blood."

(2) *Bushmen.*

(*a*) F. Shrubsall: "Crania of African Bush Races." *J. A. I.*, Vol. XXVII. 1897, pp. 263—290. Nineteen male skulls: 2 from the Transvaal, 1 from the Kalahari Desert, 7 from Cape Colony, 8 from an unknown locality and 1 Strandlooper from Port Elizabeth. These are in the British Museum, Royal College of Surgeons and other museums at Netley Hospital, Cambridge and Basel.

(*b*) F. Shrubsall: "Notes on some Bushman Crania and Bones from the South African Museum, Cape Town." *Annals of the South African Museum*, Vol. v. Part v. 1907, pp. 227—270. Measurements of 20 male skulls at Cape Town and Cambridge have been taken from this paper. Only five of these are presumed to be true Bushman, the remainder being Strandloopers. Shrubsall followed Broca's definitions in this and his other papers quoted, except that the orbital breadth is defined as O_1' in place of Broca's *Lacr.* O_1, and some additional measurements are given.

(*c*) W. Turner: "Report on the Human Crania and other Bones of the Skeletons collected during the Voyage of H.M.S. Challenger." *Challenger Reports.* Zoology, Vol. x. 1884, p. 13. One male Bushman skull from Griqualand East, one from Cape Colony and three others from an unknown locality. These specimens are now in the Edinburgh University Anatomical Museum. Broca's methods of measurement were used with a few exceptions.

(*d*) A. Hrdlička: "Catalogue of Human Crania in the United States National Museum Collections." *Proceedings of the United States National Museum*, Vol. LXXI. Art. 24, 1928, p. 107. Five male skulls in the Anatomy Department of Cape Town University. Two came from between the Fish and Kowie Rivers, one from the Cape Coast, one from Hout Bay and the other from an unknown locality.

(*e*) J. W. Spengel: *A. S. D.*, Göttingen Catalogue, 1874, p. 60. One male skull from between the Zak and Orange Rivers.

(*f*) G. Wieger: *A. S. D.*, Breslau Catalogue, 1884, p. 8. One male skull.

(*g*) R. Broom: "A Contribution to the Craniology of the Yellow-Skinned Races of South Africa." *J. R. A. I.*, Vol. LIII. 1923, pp. 132—149. Three male skulls from Cape Colony and one from Griqualand West (p. 138). The measurements given of others from the Kalahari on p. 136 could not be used as no sexing is given.

The measurements of all the male Bushman skulls taken from the above sources have been pooled to form a series which is just long enough for statistical purposes. No sub-division of the material could be profitably made. The authenticity of several

of the specimens included here, and, indeed, of nearly all the presumed Bushman crania preserved in European museums, has been questioned by Broom (*loc. cit.* pp. 133—135). Some of the skulls sent to Shrubsall labelled "Bushman" are said to be in fact Hottentot skulls, and objection is made to the use of Strandlooper material as several different races, of which the Bushman is one, are thought to have been associated with the "kitchen-midden" culture found along the coasts. It is said to be customary to suppose that any isolated skeleton found in some parts of South Africa is that of a Bushman, and most skeletons in European museums labelled as such have been accepted without any proof of their authenticity. Under these circumstances our provisional means are of very doubtful value, but they may be used on this understanding as it is not possible to obtain any more reliable ones at present. If the series we are using contains a large proportion of skulls which are not those of Bushmen, then it may be expected to show an unusually large variability and also to bear a close, though probably spurious, resemblance to a Hottentot series. It will be seen below that neither of these conditions is fulfilled.

(3) *Cameroons.* K. Drontschilow: "Metrische Studien an 93 Schädeln aus Kamerun." *A. f. A.*, Bd. XL. 1913, pp. 161—183. This paper deals with ninety-three male skulls in the Museum für Völkerkunde at Berlin from the western part of the Cameroons and representing a number of different tribes. F. von Luschan's methods of measurement are followed. In calculating the means and standard deviations we have omitted the seven specimens marked as pathological.

(4) *Congo.* R. Crewdson-Benington: "A Study of the Negro Skull with special reference to the Congo and Gaboon Crania." *Bm.* Vol. VIII. 1912, pp. 292—339. The Congo series consists of 50 male and 27 female skulls in the Museum of the Royal College of Surgeons. They represent the Batetela tribe located approximately 24° 20′ E. and 4° 51′ S.

(5) *Gaboon.* R. Crewdson-Benington. *Ibid.* Two series are considered from the Gaboon, now in the British Museum (Natural History), which were presented by Du Chaillu. The first, collected in Fernand Vaz in 1864, consists of 50 male and 44 female specimens; the second, obtained from the same place in 1880, consists of 18 male and 19 female specimens. The crude coefficient of racial likeness between the two male series is $0{\cdot}73 \pm {\cdot}17$ for 30 characters and no significant difference is found in the case of this measurement. The means obtained by pooling the two groups have been used for comparative purposes in this paper.

(6) *Galla and Somali.*

(*a*) P. Lester: "Contribution à l'Anthropologie des Somalis." *B. M. S. A. P.*, 7e Série, T. VIII. 1927, pp. 175—187. Five male Somali skulls at Paris from the Magadoxo district on the Somaliland coast.

(*b*) S. Sergi: "*Crania Habessinica.*" Rome, 1912, p. 373. Four male Galla skulls from the Witu district and three male Somali from an unknown locality. These specimens are in the Virchow collection.

(*c*) Regnault: *B. S. A. P.*, 3e Série, T. IX. 1886, p. 664. One male skull from the Somali coast in the Society's museum.

(*d*) E. T. Hamy: "Quelques observations sur l'Anthropologie des Çomalis." *B. S. A. P.*, 3e Série, T. v. 1882, pp. 697—706. One male skull from the northern Somali coast.

(*e*) P. Lester: "Étude Anthropologique des Populations de l'Éthiopie." *L'Anthropologie*, T. XXXVIII. 1928, pp. 289—315. Thirteen male Galla skulls in the Muséum d'Histoire Naturelle, Paris: one belongs to the Itu and eleven to the Arusi tribe and the other comes from the Oborrah region.

(*f*) J. Lederle: "Ein Negerschädel mit Stirnnaht, beschrieben und verglichen mit 53 anderen Negerschädeln." *A. f. A.*, Bd. VIII. 1875, pp. 177—209. Three male Galla skulls.

(*g*) R. Magnolfi: "Intorno a due crani Somali raccolti dall' Ing. Robecchi Bricchette." *A. p. A.*, Vol. LIV. 1925, pp. 137—143. Two male Somali skulls in the anatomical Museum of the University of Pavia from the Somali coast near Magadoxo.

(*h*) N. Puccioni: "Studi sui materiali e sui dati antropologici ed etnografici raccolti dalla Missione Stefanini-Paoli nella Somalia Italiana meridionale." *A. p. A.*, Vol. XLIX. 1919, pp. 41—223. Two male skulls from the Somali coast (p. 99).

(*i*) P. Paulitschke: *Beiträge zur Ethnographie und Anthropologie der Somâl, Galla und Harari*. Leipzig, 1886, p. 105. Three male Somali skulls.

The material derived from these sources is quite insufficient to establish reliable mean measurements of both the Galla and Somali. The male means were calculated for the two groups separately, however, and there was found to be a close correspondence between them. For the Galla ($\bar{n} = 15{\cdot}5$) and Somali ($\bar{n} = 14{\cdot}7$) the crude coefficient of racial likeness of $1{\cdot}12 \pm {\cdot}23$ was found for 17 characters. This appears to justify the provisional pooling of the two series and the combined means have been used for comparative purposes. If more material were available, it is probable that a racial distinction could be made between the Galla and Somali.

(7) *Hottentots.*

(*a*) F. Shrubsall: *loc. cit.* 2*a*. Nineteen male skulls at Netley Hospital, the Royal College of Surgeons, Cambridge and Basel. There are 5 from Cape Colony, 1 from the Transvaal, 2 from the Koranna tribe, 3 "from a cave in Makapan's country and probably belonging to the tribe of that name," and 8 from an unknown locality. It is said that: "The true Hottentots are found on the western and south-eastern borders of Cape Colony, but in greatest numbers and purest strain in Namaqualand and Southern Damaraland. The Griquas are a mongrel stock with a Hottentot basis inhabiting East and West Griqualand at the junction of the Orange and Vaal Rivers; and the Korannas survive in the valleys of the Upper Orange, Vaal, and Modder, and on the banks of the Central Hart River in British Bechuanaland."

(*b*) R. Broom: *loc. cit.* 2(*g*), pp. 142 and 144. Two male skulls from Upington, British Bechuanaland, and five of Korannas from Griqualand West.

(*c*) A. Hrdlička: *loc. cit.* 2(*d*), p. 110. Two male skulls in the South African Museum, Cape Town, one being from the Orange River.

The term "Hottentot" is used more vaguely than "Bushman" to cover a number of different peoples found scattered over a large area in South Africa and generally supposed to exhibit varying degrees of purity in different localities. At least ten times the craniological material at present available would be needed in order to examine satisfactorily the racial constitution of this group, and, in pooling the measurements derived from the above sources, we recognise fully the need for a more adequate treatment of the subject.

(8) *Kaffirs.*

(*a*) F. Shrubsall: *loc. cit.* 1. A series of 38 male skulls at Netley Hospital, Cambridge, the British Museum (Natural History) and the Royal College of Surgeons. Of the specimens whose origin is known, 14 come from the east of South Africa and one from Bechuanaland.

(*b*) A. Hrdlička: *loc. cit.* 2(*d*), p. 127. Six male skulls in the United States National Museum, of which five are from Port Alfred, Cape of Good Hope, and one from the Zambesi.

(*c*) E. Schmidt: *A. S. D.*, Leipzig Catalogue (1887), pp. 122—125. Thirteen male skulls, of which the origin is not stated.

(*d*) Rüdinger: *A. S. D.*, München Catalogue (1892), p. 128. One male skull (No. 552).

(*e*) R. Häcker: *A. S. D.*, Tübingen Catalogue (1902), p. 32. One male skull.

(*f*) J. Mies: *A. S. D.*, Heidelberg Catalogue (1896), p. 52. One male skull.

It is generally supposed that the term "Kaffir" has no real ethnological value as it is used loosely to cover all the Bantu negroes inhabiting the extreme south and south-east of Africa. The pooled means derived from the above sources can only be used provisionally to represent that large group of peoples among which it would almost certainly be possible to distinguish several racial types if adequate material were available. The crude coefficient of racial likeness between Shrubsall's series on the one hand ($\bar{n} = 36{\cdot}6$) and all the others ($\bar{n} = 18{\cdot}4$) is $1{\cdot}62 \pm {\cdot}20$ for 22 characters.

(9) *Negroes from Egypt.* E. Schmidt: *A. S. D.*, Leipzig Catalogue (1887), pp. 55–63. Nineteen male and 7 female skulls from Elephantine and 16 male and 13 female from Philae. These islands in the Nile are close to Assuan. No particulars relating to the origin of the series are given: they are classed as "moderne Schädel" and the series of "antike Schädel" which follow them appear to be all of dynastic date. The collection from Elephantine was made in 1875 and the skulls from Philae are said to be "aus Mooks Sammlung," the first being inscribed "Moderner Schädel

aus Philae," and all others "Antiker Schädel aus Philae." The last are presumably to be referred to the modern era. Male means of the two series and of Schmidt's modern one from Cairo are given below:

	L	*B*	100 *B*/*L*	*N* ∠	100 *NB*/*NH*
Cairo ...	176·1 (47)	137·6 (47)	78·1 (47)	66°·7 (45)	49·4 (46)
Elephantine	179·5 (19)	134·9 (19)	75·3 (19)	70°·6 (19)	55·8 (19)
Philae ...	179·9 (16)	134·0 (16)	74·5 (16)	69°·2 (15)	51·4 (15)

The nasal angles of the Elephantine and Philae, and the nasal index of the Elephantine, are decidedly higher than any others that have been given for modern or dynastic Egyptian series, and it is extremely probable that these two represent intrusive and closely related negro populations. On this assumption the means were pooled and it is shown below that some close relationships are found with true negro types.

(10) *Tanganyika Territory.* H. A. Ried: "Zur Anthropologie des abfluzlosen Rumpfschollenlandes im nordöstlichen Deutsch-Ostafrika," *Abhandlungen des Hamburgischen Kolonialinstituts,* Bd. XXXI. 1915. *Reihe B, Völkerkunde, Kulturgeschichte und Sprachen,* Bd. XVII. pp. 1—295. A collection of modern skulls was made in the north-east of German East Africa (now Tanganyika Territory) in 1911–12. Six tribes are represented, the Ssandaui (6 ♂, 4 ♀), Turu (7 ♂, 2 ♀), Burungi (10 ♂, 2 ♀), Issansu (12 ♂, 10 ♀), Kindiga (2 ♂, 1 ♀) and Irangi (1 ♂). The means for all the tribes were calculated for use in the present paper. Martin's methods of measurement were followed with a few modifications and additions (see p. 3).

(11) *Badari Egyptians.* B. N. Stoessiger: "A Study of the Badarian Crania recently excavated by the British School of Archaeology in Egypt," *Bm.*, Vol. XIX. 1927, pp. 110—150. This series of 36 male and 22 female skulls is the earliest of predynastic date that has yet been discovered in Egypt. It has been shown that the earliest series from this country approach negro types rather more closely than all later Egyptian series do. (See *Bm.*, Vol. XVII. 1925, pp. 6—9.) The Badari series has been selected for comparison with the negro types, from among a considerable number of ancient Egyptian series available, as it has the highest nasal index of all these.

(12) *Tasmanians.* G. M. Morant: "A Study of the Australian and Tasmanian Skulls, based on previously published measurements," *Bm.*, Vol. XIX. 1927, pp. 417—440. The mean measurements of Tasmanian skulls given in this paper were used for comparative purposes as they appeared to be remarkably close to our pooled means for Bushmen and Hottentots.

Fig. 1 shows the regions from which the African series dealt with in this paper were obtained, and it will be seen that there are enormous areas entirely unrepre-

sented. The material as a whole is so meagre that it hardly justifies the attempt to arrive at a preliminary classification of the races of Central and South Africa. In the case of the Kaffirs, Hottentots and Bushmen it is unlikely that our pooled means represent discrete racial types, and the Galla and Somali means probably represent two races instead of one. An interesting map showing the distribution of the cephalic index in Central Africa has been provided by Struck *. It is based principally on measurements of the living and the range for them is from 69·0 to 83·3. The highest index for the series with which we are dealing is 78·0 and for the living the corresponding value would be about 80. We have no brachycephalic negro cranial series and none for the pygmy races. From a craniometric standpoint Africa, apart from Egypt, is certainly less well known than Asia.

The male mean measurements, other than those previously published in *Biometrika*, for the material described above are given in Table IV. So few measurements are available for the female skulls that it would be unprofitable to pool them, and the Cameroons series consists of male skulls only. The female means for the Negroes from Egypt and the Tanganyika tribes are given, with the Teita values, in Table V, and the Congo and Gaboon constants are in Crewdson-Benington's paper. No comparisons have been made between the female series.

A comparison of the male variabilities of nine series with the reliable values found for the large collection of Egyptian *E* crania is made in Table VI in the case of nine characters. Standard deviations were not computed for fewer than 30 skulls. It has already been shown that the Teita and this standard Egyptian series are almost equally variable (see pp. 276—277 above), while the latter is decidedly more homogeneous than all European groups that have been described in this way. The Egyptian standard deviation of the orbital height is the smallest for this character in Table VI, but for the calvarial height the same series has the greatest variability. There are 73 σ's for negro and related types: 43 of these exceed the corresponding Egyptian values, 29 are less than they and there is equality in one case. The differences in excess of the Egyptian exceed 2·5 times their probable errors in 17 cases and in defect in 10 cases. It is interesting to note that every standard deviation for the nasal index, except the Teita, is significantly greater than the standard value with which they are here compared, while for the calvarial height that standard value is significantly greater than five others in the table. Judging from these characters, the Egyptian *E*, Teita, Congo, Gaboon, Tanganyika and the negro series from Egypt (in spite of the outstandingly high standard deviation of the cephalic index in the case of the last) have approximately equal variabilities. The Cameroons and Bushmen series are rather more variable than these and, as had been anticipated, the Galla and Somali and Kaffir collections are decidedly heterogeneous. We may conclude that the purest groups of negro skulls are as homogeneous as normal dynastic Egyptian ones and more homogeneous than any that have yet been found in Europe.

* B. Struck: "Versuch einer Karte des Kopfindex im mittleren Africa," *Zeitschrift für Ethnologie*, Bd. XLV. 1922, pp. 51—113.

TABLE IV.

Male Mean Measurements of the Teita and Related Series.

Characters	Teita *	Hottentots (pooled)	Tanganyika Territory (Ried)	Kaffirs (pooled)	Galla and Somali (pooled)	Negroes from Egypt (Elephantine and Philae) (Schmidt)	Bushmen (pooled)	Cameroons (Drontschilow)
C	1316·1 (30)	—	1299·3 (37)	1422·3 (21)	—	1351·4 (33)	—	1422·0 (83)
B′	96·1 (55)	95·2 (19)	96·5 (38)	100·1 (54)	94·7 (37)	94·8 (35)	94·1 (26)	96·9 (86)
B″	111·2 (48)	—	108·7 (37)	118·0 (50)	114·2 (25)	115·7 (35)	—	118·2 (84)
B	129·6 (54)	133·6 (28)	130·4 (38)	137·3 (58)	134·2 (36)	134·5 (35)	135·7 (54)	138·8 (84)
Bi-asterionic *B*	104·4 (51)	—	106·0 (38)	109·8 (50)	108·4 (11)	104·3 (35)	—	105·4 (65)
L	183·9 (53)	185·7 (28)	180·8 (38)	188·7 (59)	181·9 (37)	179·7 (35)	179·3 (55)	180·1 (85)
H′	130·8 (44)	131·1 (28)	129·4 (38)	136·7 (57)	133·2 (28)	133·5 (35)	126·9 (49)	135·0 (84)
H	130·3 (40)	—	—	137·8 (16)	—	135·5 (34)	—	—
OH	111·7 (48)	—	112·4 (38)	117·4 (16)	—	—	—	115·5 (84)
LB	101·5 (42)	99·6 (27)	99·8 (38)	103·7 (60)	98·9 (32)	101·2 (35)	95·2 (51)	100·1 (86)
S_1'	110·2 (52)	—	109·5 (38)	—	111·3 (10)	—	—	111·1 (84)
S_2'	113·9 (54)	—	114·4 (35)	—	113·2 (10)	—	—	112·2 (84)
S_3'	93·5 (44)	—	91·5 (35)	—	94·3 (10)	95·9 (34)	—	97·6 (85)
S_1	126·9 (52)	128·2 (17)	126·4 (38)	130·5 (39)	127·4 (11)	—	129·8 (35)	126·5 (84)
S_2	126·6 (54)	126·8 (17)	124·3 (37)	132·0 (40)	126·7 (11)	—	120·3 (35)	126·3 (84)
S_3	113·5 (44)	112·4 (17)	108·8 (35)	119·3 (39)	111·7 (11)	—	115·9 (35)	112·4 (85)
S	366·6 (42)	367·5 (17)	359·3 (38)	380·9 (52)	373·2 (18)	367·2 (34)	365·4 (40)	365·4 (84)
U	506·6 (54)	507·8 (17)	—	524·7 (53)	511·8 (16)	505·9 (35)	503·3 (28)	507·8 (85)
Glabella *U*	511·0 (50)	—	505·6 (38)	—	—	—	507·4 (5)	—
Q	301·0 (48)	—	—	314·7 (16)	—	304·6 (35)	—	—
Bregmatic *Q*	298·6 (52)	—	299·3 (38)	—	—	—	—	—
Broca's *Q*	—	293·6 (16)	—	307·2 (37)	296·3 (12)	—	292·1 (29)	—
G′H	68·5 (32)	65·9 (27)	68·3 (36)	69·4 (58)	67·7 (26)	67·4 (33)	62·1 (40)	67·9 (86)
GL	100·2 (31)	99·5 (27)	—	104·5 (56)	96·2 (25)	100·6 (34)	94·1 (38)	102·6 (86)
GB	98·1 (38)	94·3 (22)	—	—	—	97·0 (34)	93·6 (29)	99·7 (86)
J	130·9 (27)	126·0 (25)	129·1 (36)	132·9 (55)	128·2 (24)	128·7 (35)	123·8 (38)	134·2 (86)
NH, *R* or *L*	(*R*) 50·2 (47)	—	—	49·8 (16)	—	48·4 (34)	—	—

TABLE IV (*continued*).

Characters	Teita*	Hottentots (pooled)	Tanganyika Territory (Ried)	Kaffirs (pooled)	Galla and Somali (pooled)	Negroes from Egypt (Elephantine and Philae) (Schmidt)	Bushmen (pooled)	Cameroons (Drontschilow)
NH'	47·1 (39)	46·0 (26)	48·2 (37)	48·5 (42)	49·3 (28)	—	44·2 (45)	49·1 (86)
NB	27·6 (47)	26·5 (26)	26·7 (38)	27·3 (58)	25·3 (28)	25·9 (34)	25·6 (46)	27·5 (86)
DC	24·3 (23)	—	23·3 (36)	—	—	—	—	25·2 (86)
SC	9·4 (40)	—	9·7 (38)	—	—	—	—	—
O_1, *R* or *L*	(*R*) 43·6 (44)	—	(*L*) 41·1 (38)	—	—	—	—	—
O_1', *R* or *L*	(*R*) 39·7 (29)	38·2 (26)	(*L*) 39·7 (38)	—	—	—	38·2 (39)	40·9 (86)
Lacr. O_1, *R* or *L*	(*R*) 38·1 (20)	—	37·8 (36)	39·8 (51)	(*R*) 39·3 (18)	38·1 (35)	37·2 (5)	—
O_2, *R* or *L*	(*R*) 33·9 (44)	32·3 (26)	(*L*) 34·4 (38)	33·8 (59)	(*R*) 34·2 (31)	32·5 (35)	31·1 (45)	34·5 (86)
G_1'	46·9 (31)	—	—	50·1 (13)	—	50·0 (32)	—	—
G_2	39·3 (25)	—	—	41·5 (13)	—	40·8 (33)	—	—
fml	36·5 (42)	36·2 (18)	36·8 (38)	37·2 (53)	35·9 (30)	35·4 (34)	35·9 (22)	37·1 (86)
fmb	29·5 (41)	28·7 (18)	29·2 (38)	29·9 (53)	29·6 (30)	29·4 (34)	29·0 (17)	30·2 (86)
100 *B*/*L*	70·6 (53)	71·9 (28)	72·0 (36)	72·9 (57)	73·9 (36)	75·0 (35)	75·6 (54)	76·8 (84)
100 *H'*/*L*	71·4 (43)	70·6 (26)	71·6 (37)	72·4 (57)	73·1 (28)	{74·3 (35)}	70·9 (49)	74·8 (84)
100 *H*/*L*	71·2 (39)	—	—	{73·0 (16)}	—	{75·4 (34)}	—	—
100 *B*/*H'*	98·9 (44)	{101·8 (26)}	{100·8 (38)}	100·4 (55)	101·2 (27)	{100·7 (35)}	{106·9 (49)}	{102·8 (84)}
100 *B*/*H*	99·3 (40)	—	—	{99·6 (16)}	—	{99·3 (34)}	—	—
Oc. I.	59·5 (44)	—	61·1 (34)	—	64·3 (10)	—	—	65·6 (83)
100 *fmb*/*fml*	81·8 (39)	{79·3 (18)}	79·5 (38)	80·5 (52)	{82·5 (30)}	83·0 (34)	80·3 (17)	{81·4 (86)}
100 *G'H*/*GB*	69·4 (26)	69·7 (22)	—	72·3 (47)	—	70·0 (32)	66·5 (27)	{68·1 (86)}
100 *NB*/*NH*, *R* or *L*	(*R*) 55·2 (45)	—	—	56·7 (16)	—	54·0 (34)	—	—
100 *NB*/*NH'*	58·3 (36)	58·0 (26)	{55·4 (37)}	56·5 (42)	52·2 (25)	—	58·2 (45)	{56·0 (86)}
100 O_2/O_1, *R* or *L*	(*R*) 77·8 (40)	—	(*L*) {83·7 (38)}	—	—	—	—	—
100 O_2/O_1', *R* or *L*	(*R*) 85·6 (27)	84·8 (26)	(*L*) 86·6 (38)	—	—	—	81·4 (39)	{84·4 (86)}
100 O_2/*Lacr.* O_1, *R* or *L*	(*R*) 87·7 (19)	—	(*L*) {91·0 (36)}	85·4 (51)	(*R*) 86·0 (18)	85·3 (35)	81·4 (5)	—
100 G_2/G_1'	80·8 (18)	—	—	81·5 (13)	—	82·0 (31)	—	—
N ∠	69°·2 (27)	{70°·6 (27)}	—	{71°·0 (56)}	67°·2 (18)	{69°·8 (33)}	{69°·8 (38)}	{72°·0 (86)}
A ∠	71°·2 (27)	{70°·9 (27)}	—	{70°·1 (56)}	72°·1 (18)	{71°·2 (33)}	{72°·1 (38)}	{68°·9 (86)}
B ∠	39°·6 (27)	{38°·5 (27)}	—	{39°·0 (56)}	40°·8 (18)	{39°·0 (33)}	{38°·1 (38)}	{39°·1 (86)}
Alv. *P* ∠	83°·2 (30)	—	—	—	—	—	—	79°·7 (85)

* The following additional mean measurements can be given for the male Teita series: $F = 183{\cdot}7$ (53), $NH, L = 49{\cdot}6$ (51), $O_1L = 43{\cdot}4$ (48), $O_1'L = 39{\cdot}5$ (23), $O_2L = 34{\cdot}0$ (48), $G_1 = 51{\cdot}4$ (26), $EH = 12{\cdot}05$ (23), $PH = 21{\cdot}6$ (20), $DS = 10{\cdot}25$ (23), $DA = 34{\cdot}4$ (23), $SS = 2{\cdot}15$ (39), $100\,O_2/O_1, L = 78{\cdot}3$ (46), $100\,NB/NH, L = 55{\cdot}4$ (46), $100\,G_2/G_1 = 74{\cdot}0$ (16), $100\,EH/G_2 = 31{\cdot}0$ (22), $100\,DS/DC = 42{\cdot}2$ (22), $100\,SS/SC = 22{\cdot}4$ (39), Prosthion $P\angle = 81°{\cdot}3$ (27).

TABLE V.

Female Mean Measurements of the Teita and Related Series.

Characters	Teita*	Tanganyika Territory (Ried)	Negroes from Egypt (Elephantine and Philae) (Schmidt)
C	1192·4 (33)	1152·1 (17)	1220·5 (19)
B′	92·4 (63)	93·6 (18)	96·0 (20)
B″	108·3 (61)	108·0 (17)	111·3 (20)
B	126·6 (61)	127·1 (19)	130·0 (20)
Bi-asterionic *B*	101·0 (58)	102·5 (19)	100·9 (20)
L	175·7 (58)	173·2 (19)	173·8 (20)
H′	126·0 (49)	123·3 (18)	128·0 (20)
H	124·8 (47)	—	129·9 (20)
OH	107·2 (51)	107·2 (18)	109·5 (20)
LB	96·6 (46)	94·3 (17)	96·3 (20)
S_1'	106·3 (59)	105·5 (18)	—
S_2'	109·4 (61)	109·7 (17)	—
S_3'	90·4 (51)	88·2 (16)	92·3 (20)
S_1	123·1 (59)	122·7 (18)	—
S_2	122·1 (61)	121·9 (18)	—
S_3	108·2 (51)	104·5 (16)	—
S	353·6 (48)	350·9 (18)	352·8 (20)
U	488·7 (59)	—	488·3 (20)
Glabella *U*	488·0 (52)	485·9 (19)	—
Q′	289·4 (51)	—	293·5 (20)
Bregmatic *Q′*	287·4 (57)	289·2 (18)	—
G′H	63·9 (32)	63·9 (14)	63·5 (20)
GL	94·9 (32)	—	94·7 (20)
GB	94·1 (41)	—	92·8 (18)
J	122·9 (35)	122·9 (35)	120·9 (19)
NH, *R* or *L*	(*R*) 47·0 (50)	—	46·1 (20)
NH′	44·0 (45)	44·6 (16)	—
NB	27·0 (48)	26·1 (18)	24·7 (20)
DC	23·7 (24)	23·0 (18)	—
SC	10·3 (51)	9·4 (17)	—
O_1, *R* or *L*	(*R*) 42·0 (51)	(*L*) 39·1 (18)	—
O_1', *R* or *L*	(*R*) 38·1 (32)	(*L*) 38·2 (18)	—
Lacr. O_1, *R* or *L*	(*R*) 37·0 (14)	(*L*) 36·1 (18)	37·5 (20)
O_2, *R* or *L*	(*R*) 33·3 (49)	(*L*) 33·1 (18)	32·3 (20)
G_1'	44·7 (34)	—	45·3 (18)
G_2	36·9 (26)	—	38·5 (17)
fml	35·6 (46)	34·4 (18)	35·3 (20)
fmb	28·6 (50)	27·1 (18)	28·3 (20)
100 *B*/*L*	72·0 (56)	73·4 (18)	74·7 (20)
100 *B′*/*L*	71·6 (48)	71·6 (17)	{73·7 (20)}
100 *H*/*L*	71·2 (47)	—	{74·7 (20)}
100 *B*/*H′*	100·3 (49)	{103·1 (19)}	{101·6 (20)}
100 *B*/*H*	101·0 (47)	—	{100·1 (20)}
Oc. I.	60·7 (51)	61·7 (16)	—
100 *fmb*/*fml*	80·2 (45)	79·0 (17)	80·2 (20)
100 *G′H*/*GB*	68·6 (26)	—	68·6 (18)
100 *NB*/*NH*, *R* or *L*	(*R*) 57·5 (44)	—	53·9 (20)
100 *NB*/*NH′*	61·8 (41)	{58·5 (16)}	—
100 O_2/O_1, *R* or *L*	(*R*) 79·3 (48)	(*L*) {84·7 (18)}	—
100 O_2/O_1', *R* or *L*	(*R*) 88·4 (24)	(*L*) 86·5 (18)	—
100 O_2/*Lacr.* O_1, *R* or *L*	(*R*) 89·1 (14)	(*L*) {91·7 (18)}	86·2 (20)
100 G_2/G_1G_1'	84·6 (18)	—	85·2 (13)
N ∠	68°·9 (29)	—	{69°·1 (20)}
A ∠	72°·1 (29)	—	{72°·1 (20)}
B ∠	39°·0 (29)	—	{38°·8 (20)}
Alv. *P* ∠	83°·8 (31)	—	—

* The following additional mean measurements can be given for the female Teita series: *F*=176·4 (59), *NH*, *L*=47·0 (48), O_1L=41·7 (49), $O_1'L$=37·9 (26), O_2L=33·25 (50), G_1=48·4 (32), *EH*=10·9 (21), *PH*=19·6 (22), *DS*=9·4 (23), *DA*=33·5 (23), *SS*=1·92 (49), 100 O_2/O_1, *L*=79·6 (49), 100 *NB*/*NH*, *L*=57·6 (44), 100 G_2/G_1=78·1 (16), 100 *EH*/G_2=29·4 (21) 100 *DS*/*DC*=39·9 (23), 100 *SS*/*SC*=18·6 (49), Prosthion *P* ∠ =81°·5 (33).

TABLE VI.

Male Standard Deviations of the Egyptian E and other African Series (all for 30 or more skulls).

	L	*B*	*H'**	*G'H*	O_2	*U**	100 *B/L*	100 *H'/L**	100 *NB/NH**
Egyptian *E* (26th—30th Dynasties)	5·72±·09	4·76±·08	5·03±·08	4·15±·07	1·88±·03	13·77± ·22	2·68±·06	2·94±·05	3·82±·06
Teita	6·07±·40	4·33±·28	4·47±·32	4·46±·38	1·97±·14	13·51± ·88	3·01±·20	2·53±·18	3·74±·26
Congo (Crewdson-Benington)	7·05±·49	5·00±·34	4·13±·28	4·14±·28	2·19±·15	17·24±1·16	2·58±·17	2·88±·20	4·91±·33
Cameroons (Drontschilow) ...	5·88±·30	4·76±·25	4·35±·23	4·45±·23	1·90±·10	13·82± ·71	3·04±·16	3·08±·16	4·63±·24
Gaboon, 1864 (Benington) ...	5·61±·38	3·38±·23	5·02±·34	6·01±·41	2·09±·14	11·44± ·77	2·77±·19	2·43±·16	4·89±·33
Negroes from Egypt (Schmidt)	5·66±·46	5·93±·48	3·83±·31	3·84±·32	2·20±·18	10·01± ·81	4·49±·36	—	5·58±·46
Tanganyika Territory (Ried)	5·63±·44	4·70±·36	4·03±·31	4·79±·38	2·16±·16	15·60±1·21	2·21±·18	2·60±·20	—
Bushmen (pooled)	5·64±·36	4·85±·31	4·06±·28	4·84±·36	2·03±·14	—	3·42±·22	2·20±·15	6·51±·46
Galla and Somali (pooled) ...	6·75±·53	6·35±·51	—	—	2·77±·24	—	3·77±·30	—	—
Kaffirs (pooled)	7·12±·44	5·08±·32	4·55±·29	4·92±·31	2·25±·14	14·06± ·92	3·70±·23	2·53±·16	5·21±·38

* Some of these standard deviations are for *H* in place of *H'*, Glabella *U* in place of *U*, 100 *H/L* in place of 100 *H'/L* and 100 *NB/NH'* in place of 100 *NB/NH*.

5. *Comparisons by the Method of the Coefficient of Racial Likeness.* With the usual notation, the forms of the coefficient used in this paper are:

$$(a)\ \text{Crude: C.R.L.} = \frac{1}{m}\Sigma\frac{(M_s - M_s')^2}{\sigma_s^2} \times \frac{n_s n_s'}{n_s + n_s'} - 1 \pm \cdot 67449\sqrt{\frac{2}{m}}$$

$$= \frac{1}{m}\Sigma(a) - 1 \pm \cdot 67449\sqrt{\frac{2}{m}}.$$

$$(b)\ \text{Reduced: C.R.L.} = 50 \times \frac{\bar{n}_1 + \bar{n}_2}{\bar{n}_1\bar{n}_2}\left\{\frac{1}{m}\Sigma(a) - 1\right\} \pm 50 \times \frac{\bar{n}_1 + \bar{n}_2}{\bar{n}_1\bar{n}_2}\left\{\cdot 67449\sqrt{\frac{2}{m}}\right\}.$$

The reduced coefficient is intended to give a good approximation to the value which the crude coefficient would have if all means used of both sexes in the comparison had been based on 100 individuals. The effect that the varying sizes of the samples has on the values of the crude coefficients is thus compensated for, and direct comparisons can be made between the reduced values. These are used, in default of any more exact method of comparison, as though they were measures of the absolute divergencies of the types. It has been shown above that the variabilities of the majority of the series used are approximately the same and close to those of the long Egyptian *E* series. Standard deviations of the last were hence used in calculating all the coefficients.

The crude coefficients between all pairs of the 13 series selected are given in Table VII for all available of the 31 characters and for the indices and angles alone. There is, in general, a fairly close correspondence between the two kinds of comparison, but a number of notable exceptions to this rule may be found. Among the 78 comparisons there are 14 for which the coefficients for all characters more than double the corresponding ones for indices and angles alone, and in eight of these cases the Kaffir is one of the series involved. The Tanganyika and Kaffir means have a crude coefficient of $14{\cdot}30 \pm {\cdot}22$ based on 19 characters, but this falls to the insignificant value of $0{\cdot}33 \pm {\cdot}43$ when the 5 indices are considered alone. It will be shown later that, when compared with other African series, the Kaffir type is unusually large and it is its size which distinguishes it rather than its shape. There are two examples in Table VII of the coefficient for indices and angles being twice the corresponding ones for all characters. The disadvantage in practice of using the measurements of shape alone is that few of them are available and that the number differs considerably in different comparisons.

The reduced coefficients for all characters are given in Table VIII, and the best classification of the material which can be made at present may be deduced from these. The lowest value found is that of $4{\cdot}26 \pm {\cdot}52$ between the Teita and Tanganyika series, so every one is clearly significant. Although one Egyptian and the Tasmanian types are compared with all the other series used, the highest reduced coefficient is found not with one of these, but between the neighbouring Bushmen and Kaffirs ($81{\cdot}81 \pm {\cdot}44$). Similar comparisons between European series have led to a considerable number of reduced values less than four and several greater than 100, so none of the present comparisons indicates either very close similarity, or very

		Congo (48·5)*	Cameroons (85·4)	Gaboon (66·5)
Congo (48·5)*	All characters	—	10·34± ·20 (24)	4·41± ·18 (29)
	Indices and Angles		13·40± ·32 (9)	5·33± ·29 (11)
Cameroons (85·4)	All characters	10·34± ·20 (24)	—	4·79± ·20 (23)
	Indices and Angles	13·40± ·32 (9)		6·18± ·34 (8)
Gaboon (66·5)	All characters	4·41± ·18 (29)	4·79± ·20 (23)	—
	Indices and Angles	5·33± ·29 (11)	6·18± ·34 (8)	
Negroes from Egypt (34·6)	All characters	5·43± ·18 (27)	7·37± ·20 (22)	2·45± ·19 (26)
	Indices and Angles	4·78± ·32 (9)	5·30± ·36 (7)	0·40± ·32 (9
Angoni (24·3)	All characters	15·94± ·22 (19)	7·94± ·20 (23)	4·53± ·22 (19
	Indices and Angles	17·19± ·36 (7)	9·94± ·32 (9)	4·67± ·36 (7
Kaffirs (54·2)	All characters	23·48± ·19 (26)	15·01± ·20 (24)	12·88± ·18 (27
	Indices and Angles	20·83± ·32 (9)	15·76± ·34 (8)	8·36± ·32 (9
Galla and Somali (28·5)	All characters	7·21± ·21 (20)	9·13± ·20 (22)	4·76± ·21 (20
	Indices and Angles	11·41± ·36 (7)	12·09± ·34 (8)	7·75± ·36 (
Tanganyika (37·5)	All characters	20·61± ·21 (20)	16·31± ·20 (22)	16·73± ·22 (1
	Indices and Angles	52·45± ·43 (5)	23·36± ·36 (7)	36·75± ·43 (
Teita (40·7)	All characters	26·10± ·17 (31)	21·29± ·18 (28)	19·44± ·17 (3
	Indices and Angles	40·03± ·28 (12)	34·97± ·29 (11)	2·41± ·29 (1
Hottentots (24·3)	All characters	12·14± ·21 (20)	14·70± ·20 (24)	9·37± ·21 (3
	Indices and Angles	21·92± ·36 (7)	13·64± ·32 (9)	13·29± ·36
Bushmen (41·0)	All characters	14·52± ·21 (20)	31·53± ·20 (24)	24·54± ·21 (3
	Indices and Angles	16·90± ·36 (7)	15·19± ·32 (9)	20·87± ·36
Badari Egyptians (34·3)	All characters	18·22± ·17 (31)	29·82± ·18 (28)	19·86± ·18 (
	Indices and Angles	28·20± ·28 (12)	31·37± ·29 (11)	28·17± ·30 (
Tasmanians (48·1)	All characters	15·70± ·21 (20)	13·43± ·20 (24)	17·09± ·21 (
	Indices and Angles	21·59± ·39 (6)	13·38± ·34 (8)	13·99± ·39

* The numbers in brackets following the designation of the series give the

TABLE VII.

Crude Male Coefficients of Racial Likeness between Negro and Related Series.

Gaboon (66·5)	Negroes from Egypt (34·6)	Angoni (24·3)	Kaffirs (54·2)	Galla and Somali (28·5)	Tanganyika (37·5)	
4·41 ± ·18 (29) 5·33 ± ·29 (11)	5·43 ± ·18 (27) 4·78 ± ·32 (9)	15·94 ± ·22 (19) 17·19 ± ·36 (7)	23·48 ± ·19 (26) 20·83 ± ·32 (9)	7·21 ± ·21 (20) 11·41 ± ·36 (7)	20·61 ± ·21 (20) 52·45 ± ·43 (5)	26·1 40·0
4·79 ± ·20 (23) 6·18 ± ·34 (8)	7·37 ± ·20 (22) 5·30 ± ·36 (7)	7·94 ± ·20 (23) 9·94 ± ·32 (9)	15·01 ± ·20 (24) 15·76 ± ·34 (8)	9·13 ± ·20 (22) 12·09 ± ·34 (8)	16·31 ± ·20 (22) 23·36 ± ·36 (7)	21·2 34·9
—	2·45 ± ·19 (26) 0·40 ± ·32 (9)	4·53 ± ·22 (19) 4·67 ± ·36 (7)	12·88 ± ·18 (27) 8·36 ± ·32 (9)	4·76 ± ·21 (20) 7·75 ± ·36 (7)	16·73 ± ·22 (19) 36·75 ± ·43 (5)	19·4 2·4
2·45 ± ·19 (26) 0·40 ± ·32 (9)	—	3·51 ± ·22 (19) 2·51 ± ·36 (7)	10·03 ± ·18 (28) 3·07 ± ·30 (10)	1·34 ± ·21 (21) 1·02 ± ·36 (7)	5·77 ± ·23 (17) 10·00 ± ·48 (4)	5·7 8·1
4·53 ± ·22 (19) 4·67 ± ·36 (7)	3·51 ± ·22 (19) 2·51 ± ·36 (7)	—	2·59 ± ·21 (21) 1·08 ± ·34 (8)	5·80 ± ·21 (20) 6·80 ± ·36 (7)	5·77 ± ·22 (19) 3·95 ± ·39 (6)	4·1 4·3
12·88 ± ·18 (27) 8·36 ± ·32 (9)	10·03 ± ·18 (28) 3·07 ± ·30 (10)	2·59 ± ·21 (21) 1·08 ± ·34 (8)	—	9·84 ± ·20 (23) 5·20 ± ·34 (8)	14·30 ± ·22 (19) 0·33 ± ·43 (5)	11·4 3·7
4·76 ± ·21 (20) 7·75 ± ·36 (7)	1·34 ± ·21 (21) 1·02 ± ·36 (7)	5·80 ± ·21 (20) 6·80 ± ·36 (7)	9·84 ± ·20 (23) 5·20 ± ·34 (8)	—	3·90 ± ·22 (19) 4·93 ± ·39 (6)	7·1 10·5
16·73 ± ·22 (19) 36·75 ± ·43 (5)	5·77 ± ·23 (17) 10·00 ± ·48 (4)	5·77 ± ·22 (19) 3·95 ± ·39 (6)	14·30 ± ·22 (19) 0·33 ± ·43 (5)	3·90 ± ·22 (19) 4·93 ± ·39 (6)	—	1·6 3·1
19·44 ± ·17 (30) 2·41 ± ·29 (11)	5·76 ± ·18 (29) 8·17 ± ·32 (9)	4·16 ± ·20 (23) 4·30 ± ·32 (9)	11·48 ± ·18 (28) 3·78 ± ·30 (10)	7·15 ± ·20 (24) 10·52 ± ·32 (9)	1·63 ± ·20 (23) 3·14 ± ·36 (7)	
9·37 ± ·21 (20) 13·29 ± ·36 (7)	3·34 ± ·21 (20) 6·32 ± ·36 (7)	4·12 ± ·20 (23) 4·20 ± ·32 (9)	8·96 ± ·20 (22) 1·45 ± ·34 (8)	4·57 ± ·20 (22) 8·18 ± ·36 (7)	3·59 ± ·21 (20) 0·96 ± ·39 (6)	2·7 0·9
24·54 ± ·21 (20) 20·87 ± ·36 (7)	10·48 ± ·21 (20) 10·68 ± ·36 (7)	19·42 ± ·20 (23) 16·82 ± ·32 (9)	37·49 ± ·20 (22) 15·56 ± ·34 (8)	13·45 ± ·21 (21) 13·00 ± ·36 (7)	18·06 ± ·21 (20) 18·27 ± ·39 (6)	20·8 29·3
19·86 ± ·18 (29) 28·17 ± ·30 (10)	6·24 ± ·18 (27) 6·99 ± ·32 (9)	9·14 ± ·20 (23) 8·67 ± ·32 (9)	21·17 ± ·18 (29) 10·01 ± ·30 (10)	4·33 ± ·20 (22) 2·90 ± ·34 (8)	8·16 ± ·20 (22) 3·23 ± ·36 (7)	8·9 4·3
17·09 ± ·21 (20) 13·99 ± ·39 (6)	6·79 ± ·21 (20) 8·47 ± ·36 (7)	10·31 ± ·20 (22) 11·67 ± ·34 (8)	19·72 ± ·20 (22) 6·45 ± ·36 (7)	11·05 ± ·20 (22) 15·84 ± ·36 (7)	8·87 ± ·21 (21) 12·29 ± ·36 (7)	11·6 18·8

eries give the mean number of skulls available for the characters used in computing the coefficients in the case of the comparison invo

s.

ayika (5)	Teita (40·7)	Hottentots (24·3)	Bushmen (41·0)	Badari Egyptians (34·3)	Tasmanians (48·1)
·21 (20) ·43 (5)	26·10 ± ·17 (31) 40·03 ± ·28 (12)	12·14 ± ·21 (20) 21·92 ± ·36 (7)	14·52 ± ·21 (20) 16·90 ± ·36 (7)	18·22 ± ·17 (31) 28·20 ± ·28 (12)	15·70 ± ·21 (20) 21·59 ± ·39 (6)
·20 (22) ·36 (7)	21·29 ± ·18 (28) 34·97 ± ·29 (11)	14·70 ± ·20 (24) 13·64 ± ·32 (9)	31·53 ± ·20 (24) 15·19 ± ·32 (9)	29·82 ± ·18 (28) 31·37 ± ·29 (11)	13·43 ± ·20 (24) 13·38 ± ·34 (8)
·22 (19) ·43 (5)	19·44 ± ·17 (30) 2·41 ± ·29 (11)	9·37 ± ·21 (20) 13·29 ± ·36 (7)	24·54 ± ·21 (20) 20·87 ± ·36 (7)	19·86 ± ·18 (29) 28·17 ± ·30 (10)	17·09 ± ·21 (20) 13·99 ± ·39 (6)
·23 (17) ·48 (4)	5·76 ± ·18 (29) 8·17 ± ·32 (9)	3·34 ± ·21 (20) 6·32 ± ·36 (7)	10·48 ± ·21 (20) 10·68 ± ·36 (7)	6·24 ± ·18 (27) 6·99 ± ·32 (9)	6·79 ± ·21 (20) 8·47 ± ·36 (7)
·22 (19) ·39 (6)	4·16 ± ·20 (23) 4·30 ± ·32 (9)	4·12 ± ·20 (23) 4·20 ± ·32 (9)	19·42 ± ·20 (23) 16·82 ± ·32 (9)	9·14 ± ·20 (23) 8·67 ± ·32 (9)	10·31 ± ·20 (22) 11·67 ± ·34 (8)
·22 (19) ·43 (5)	11·48 ± ·18 (28) 3·78 ± ·30 (10)	8·96 ± ·20 (22) 1·45 ± ·34 (8)	37·49 ± ·20 (22) 15·56 ± ·34 (8)	21·17 ± ·18 (29) 10·01 ± ·30 (10)	19·72 ± ·20 (22) 6·45 ± ·36 (7)
·22 (19) ·39 (6)	7·15 ± ·20 (24) 10·52 ± ·32 (9)	4·57 ± ·20 (22) 8·18 ± ·36 (7)	13·45 ± ·21 (21) 13·00 ± ·36 (7)	4·33 ± ·20 (22) 2·90 ± ·34 (8)	11·05 ± ·20 (22) 15·84 ± ·36 (7)
-	1·63 ± ·20 (23) 3·14 ± ·36 (7)	3·59 ± ·21 (20) 0·96 ± ·39 (6)	18·06 ± ·21 (20) 18·27 ± ·39 (6)	8·16 ± ·20 (22) 3·23 ± ·36 (7)	8·87 ± ·21 (21) 12·29 ± ·36 (7)
·20 (23) ·36 (7)	—	2·78 ± ·20 (24) 0·96 ± ·32 (9)	20·83 ± ·20 (24) 20·39 ± ·32 (9)	8·92 ± ·17 (31) 4·57 ± ·28 (12)	11·61 ± ·20 (24) 18·80 ± ·34 (8)
·21 (20) ·39 (6)	2·78 ± ·20 (24) 0·96 ± ·32 (9)	—	5·90 ± ·19 (25) 7·30 ± ·32 (9)	4·79 ± ·20 (24) 8·27 ± ·32 (9)	4·34 ± ·20 (24) 5·76 ± ·34 (8)
·21 (20) ·39 (6)	20·83 ± ·20 (24) 29·39 ± ·32 (9)	5·90 ± ·19 (25) 7·30 ± ·32 (9)	—	13·93 ± ·20 (24) 22·15 ± ·32 (9)	8·48 ± ·20 (23) 5·89 ± ·34 (8)
·20 (22) ·36 (7)	8·92 ± ·17 (31) 4·57 ± ·28 (12)	4·79 ± ·20 (24) 8·27 ± ·32 (9)	13·93 ± ·20 (24) 22·15 ± ·32 (9)	—	19·62 ± ·20 (24) 26·49 ± ·34 (8)
·21 (21) ·36 (7)	11·61 ± ·20 (24) 18·80 ± ·34 (8)	4·34 ± ·20 (24) 5·76 ± ·34 (8)	8·48 ± ·20 (23) 5·89 ± ·34 (8)	19·62 ± ·20 (24) 26·49 ± ·34 (8)	—

e comparison involving the largest number of these characters for the particular series.

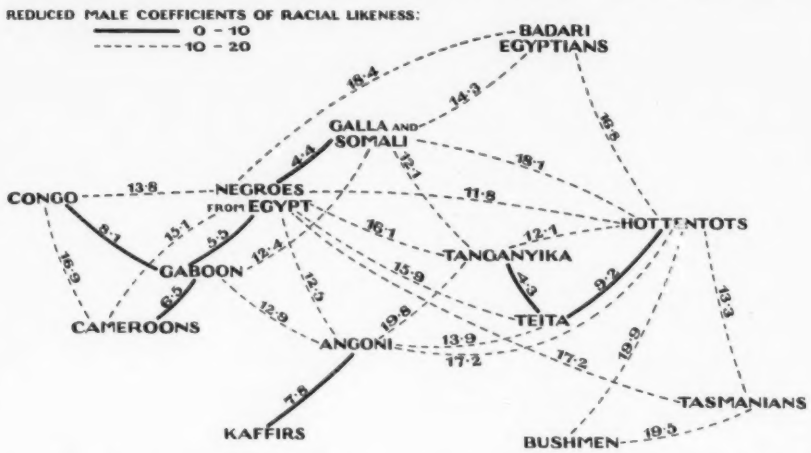
TABLE VIII.

Reduced Male Coefficients of Racial Likeness between Negro and Related Series.

	Congo	Cameroons	Gaboon	Negroes from Egypt	Angoni	Kaffirs	Galla and Somali	Tanganyika	Teita	Hottentots	Bushmen	Badari Egyptians	Tasmanians
Congo	—	16·93 ± ·32	8·07 ± ·32	13·82 ± ·47	49·93 ± ·69	50·61 ± ·40	20·89 ± ·62	49·27 ± ·51	60·70 ± ·40	38·13 ± ·67	33·74 ± ·49	46·33 ± ·44	33·35 ± ·45
Cameroons	16·93 ± ·32	—	6·48 ± ·27	15·09 ± ·42	21·46 ± ·54	23·48 ± ·31	21·22 ± ·50	31·44 ± ·39	39·77 ± ·34	38·87 ± ·52	58·27 ± ·36	61·34 ± ·37	22·22 ± ·32
Gaboon	8·07 ± ·32	6·48 ± ·27	—	5·48 ± ·42	12·93 ± ·63	23·92 ± ·34	12·42 ± ·56	35·00 ± ·46	39·06 ± ·35	26·70 ± ·61	49·87 ± ·43	44·51 ± ·40	31·31 ± ·39
Negroes from Egypt	13·82 ± ·47	15·09 ± ·42	5·48 ± ·42	—	12·48 ± ·78	26·22 ± ·49	4·41 ± ·69	16·10 ± ·64	15·94 ± ·49	11·84 ± ·76	28·65 ± ·58	18·38 ± ·54	17·21 ± ·54
Angoni	49·93 ± ·69	21·46 ± ·54	12·93 ± ·63	12·48 ± ·78	—	7·82 ± ·63	22·28 ± ·82	19·76 ± ·75	13·93 ± ·67	17·23 ± ·83	65·61 ± ·67	32·67 ± ·71	32·43 ± ·64
Kaffirs	50·61 ± ·40	23·48 ± ·31	23·92 ± ·34	26·22 ± ·49	7·82 ± ·63	—	25·69 ± ·52	32·67 ± ·50	27·17 ± ·43	26·86 ± ·61	81·81 ± ·44	54·44 ± ·45	39·40 ± ·41
Galla and Somali	20·89 ± ·62	21·22 ± ·50	12·42 ± ·56	4·41 ± ·69	22·28 ± ·82	25·69 ± ·52	—	12·07 ± ·68	22·56 ± ·62	18·06 ± ·80	40·87 ± ·63	14·26 ± ·67	32·05 ± ·59
Tanganyika	49·27 ± ·51	31·44 ± ·39	35·00 ± ·46	16·10 ± ·64	19·76 ± ·75	32·67 ± ·50	12·07 ± ·68	—	4·26 ± ·52	12·11 ± ·72	46·12 ± ·54	22·83 ± ·57	21·09 ± ·49
Teita	60·70 ± ·40	39·77 ± ·34	39·06 ± ·35	15·94 ± ·49	13·93 ± ·67	27·17 ± ·43	22·56 ± ·62	4·26 ± ·52	—	9·19 ± ·64	52·22 ± ·49	24·54 ± ·47	26·80 ± ·45
Hottentots	38·13 ± ·67	38·87 ± ·52	26·70 ± ·61	11·84 ± ·76	17·23 ± ·83	26·86 ± ·61	18·06 ± ·80	12·11 ± ·72	9·19 ± ·64	—	19·88 ± ·64	16·84 ± ·69	13·27 ± ·59
Bushmen	33·74 ± ·49	58·27 ± ·36	49·87 ± ·43	28·65 ± ·58	65·61 ± ·67	81·81 ± ·44	40·87 ± ·63	46·12 ± ·54	52·22 ± ·49	19·88 ± ·64	—	37·96 ± ·53	19·50 ± ·46
Badari Egyptians	46·33 ± ·44	61·34 ± ·37	44·51 ± ·40	18·38 ± ·54	32·67 ± ·71	54·44 ± ·45	14·26 ± ·67	22·83 ± ·57	24·54 ± ·47	16·84 ± ·69	37·96 ± ·53	—	49·55 ± ·49
Tasmanians	33·35 ± ·45	22·22 ± ·32	31·31 ± ·39	17·21 ± ·54	32·43 ± ·64	39·40 ± ·41	32·05 ± ·59	21·09 ± ·49	26·80 ± ·45	13·27 ± ·59	19·50 ± ·46	49·55 ± ·49	—

distant relationship. The male value for the contemporary London series from Farringdon Street and Whitechapel is 3·83 ± ·19. Comparisons between other groups of races have shown that the most suggestive and consistent classification is reached by considering only the closest degrees of affinity, while neglecting almost entirely the different degrees of remoter resemblance. Fig. 2 shows the arrangement suggested by the reduced coefficients less than 20. It is always found in making comparisons between a number of series that it is impossible to arrange them in

FIG. 2. THE INTER-RELATIONSHIPS OF AFRICAN NEGRO AND ALLIED RACES.



two dimensions so that their distances apart are even roughly proportional to all their closest degrees of relationship measured in this way. Considering only the reduced coefficients less than 10, division can be made into three groups:

(*a*) Congo, Cameroons, Gaboon, Negroes from Egypt, Galla and Somali. The first three of these are from West Africa, which is generally supposed to be the home of the true negro; the Egyptian series probably came from the Sudan; and the Galla and Somali are usually thought to be essentially "Hamitic" in physical type. The first three, and possibly the fourth, represent Bantu-speaking peoples, but the Galla and Somali speak an Hamitic language.

(*b*) Kaffirs and Angoni. The physical similarity of these two southern Bantu-speaking peoples is not surprising.

(*c*) Tanganyika, Teita, and Hottentots. The close resemblance between the groups from Tanganyika Territory and Kenya Colony is to be expected from cultural evidence and from their geographical position, but it is surprising to find that they are linked up with the non-Bantu Hottentots, and that the last bear their closest resemblance to the Teita who are geographically further removed from them than are the peoples of Tanganyika Territory.

When the reduced coefficients less than 10 are considered alone, the Bushmen, Badari Egyptians, and Tasmanians have no connections with the other series. The division of the latter into the three groups distinguished above does not accord well with the generally accepted classification of the races of Africa (based on the very imperfect evidence of a few measurements on the living, such as the stature, the cephalic and nasal indices, and on characters of the hair and skin), or with cultural and linguistic groupings. If we take into account a rather more distant degree of relationship—say a reduced coefficient less than 14—the divisions between the closely inter-related groups break down; for instance, the Angoni join up with both the West and East African negro types, the Hottentots become connected with the Negroes from Egypt, and the Galla and Somali with the Tanganyika tribes. The system is now, as it were, a continuous one embracing all the Bantu races represented and also the non-Bantu Hottentots and Galla and Somali. The predynastic Egyptians and the Bushmen are still excluded, but there is an unexpected link between the Hottentots and Tasmanians. Once more raising the upper limit to include all reduced coefficients less than 20, the Badari series joins up with the two geographically nearest to it (Negroes from Egypt and Galla and Somali) and also, unexpectedly, with the more distant Hottentots. The Tasmanians acquire a connection with the Negroes from Egypt and the Bushmen are just entitled to be linked up with the Hottentots and Tasmanians. It is thus clear that no sharp distinction can be made between Western, Eastern and Southern negro races, and it is probable that the divisions between them would become less clear than they are here shown to be if more types were available for comparison.

It must be admitted that there are several unexpected features of this classification which has been reached by purely statistical means. There is no close correspondence between the affinities of the types and their geographical positions. The Congo and Cameroons series may be supposed to represent the most typical West African races, but they are connected with those of East and South Africa by the Gaboon series which came from a district 800 miles further *west* than that from which the Congo crania were obtained. Kenya Colony lies to the north of Tanganyika Territory, but the Teita have closer relationships to the southern Angoni and Hottentots, while the Tanganyika tribes resemble more closely the Negroes from Egypt and the Galla and Somali. The suggested relationships of the Hottentots would certainly not have been expected. It must be noted that the Bushmen and Hottentot series are less well authenticated than the others, but they are clearly differentiated from each other and still more clearly from the Kaffirs. The position of the Tasmanians is, however, the most surprising. They share with the Bushmen the distinction of being supposed a highly specialised type, though the remote relationship between Bushmen and Hottentots and their linking up with the negroes through the Hottentots is generally accepted. The Tasmanians have a male reduced coefficient of $21{\cdot}88 \pm {\cdot}29$ with the A-type Australians, no closer connection having been previously found for them. But now a value of 13·3 has been found between them and the Hottentots, and values less than 20 with the Negroes from Egypt and with the Bushmen. All the lowest reduced coefficients

that have yet been found with the Tasmanians and Bushmen are shown in Fig. 2, and they are of a higher order than the lowest found for any of the other series in the diagram. Every negro series has at least one reduced coefficient less than 9, and the Badari can be connected up with two other predynastic Egyptian series by values less than 5. It has been shown that the earlier Egyptian types approach negro series rather more closely than the later Egyptian types do, but the earliest and most negroid of all—the Badari—is still quite widely removed from all true negro races. The Bushmen, and to a lesser extent the Tasmanians, are found by these means to be the most specialised of the types with which we are dealing. The most generalised types appear to be the Negroes from Egypt, with 10 reduced coefficients less than 20, and the Hottentots with 8 of that order. The position of the Kaffirs is peculiar as it has only one of these connections, though that one indicates a close degree of resemblance.

The present classification is only claimed to be a preliminary one, and it should not be rejected merely because it does not accord closely with the generally accepted theories of the relationships of the African races. These theories have been based almost entirely on very inadequate data obtained from the living populations. The material used in the present paper is also inadequate, but the use of purely quantitative methods applied to cranial measurements, which have many advantages over those of the living, appears to offer quite the most hopeful approach for future research in this direction. The most pressing need is for more and, if possible, longer series of crania of negroes, Bushmen, and Hottentots.

6. *Comparison of Single Measurements.* The classification suggested by the coefficients of racial likeness has been discussed, and it may be asked whether it would have been possible to arrive at similar results by considering the characters separately and balancing the evidence which they provide. Such an enquiry is not likely to be profitable unless it is known whether the differences between the means are statistically significant or not. The α's found in computing the coefficients enable us to decide this question at once, and it will be sufficient to consider only the 31 characters compared in that way. It is supposed that a difference is very unlikely to have arisen as the result of random sampling from the same population if the α is greater than 10. As has been found in the case of inter-comparisons between other groups of races, there are marked differences between the characters in so far as they are able to discriminate our 13 negro and related series. In the case of the palatal index 16 comparisons could be made and not a single one of these is significant. The foraminal breadth is available for all the series, and of the 78 comparisons made there is only one which probably indicates true racial distinction. Such measurements are obviously quite useless for our purpose and we may suppose that all others will be, too, if fewer than 30 per cent. of their α's are greater than 10. The other characters which must be discarded for this reason are, in ascending order of that percentage: 100 fmb/fml, fml, U, $A\angle$, 100 $G'H/GB$, S, G_2, C, and NH'. The palatal length, transverse arc, and profile angle have a larger percentage of significant differences, but fewer than 17 α's could be found for each one of these measure-

TABLE IX.

Comparisons of Single Measurements which show most Significant Differences: Male Means.

		Oc. I.	100 *B/L*	*J*	O_2	*B*	O_1'	100 *H'/L*	*N* ∠	*G'H*	*NB*	100 *NB/NH'*	*H'*	100 *B/H'*	*L*	*B'*	100 O_2/O_1'	*LB*
Number of Comparisons		21	78	66	78	78	40*	78	66	78	78	57†	78	78	78	78	40*	78
Percentage of *a*'s > 10		61·9	57·7	56·1	52·6	48·7	47·5	43·6	42·4	42·3	41·0	38·6	37·2	37·2	35·9	35·9	35·0	34·6
West African Races	Gaboon	67·8	75·6	130·0	35·1	136·2	—	75·4	71°·0	66·6	26·7	—	135·7	100·4	180·2	97·0	—	100·7
	Cameroons ...	65·6	76·8	134·2	34·5	138·8	40·9	74·8	72°·0	67·9	27·5	56·0	135·0	102·8	180·1	96·9	84·4	100·1
	Congo	68·8	78·0	126·5	34·6	138·5	—	75·4	69°·3	63·4	26·0	—	133·8	103·6	177·9	97·5	—	98·2
East and South African Races (excluding Bushmen)	Negroes from Egypt	—	75·0	128·7	32·5	134·5	—	74·3	69°·8	67·4	25·9	—	133·5	100·7	179·7	94·8	—	101·2
	Galla and Somali	—	73·9	128·2	34·2	134·2	—	73·1	67°·2‡	67·7	25·3	52·2	133·2	101·2	181·9	94·7	—	98·9
	Angoni	—	73·1	—	33·3	134·6	38·0	74·3	70°·5	70·3	28·0	58·0	137·0	98·2	184·3	98·0	88·1	102·5
	Kaffirs	—	72·9	132·9	33·8	137·3	—	72·4	71°·0	69·4	27·3	56·5	136·7	100·4	188·7	100·1	—	103·7
	Tanganyika Territory	61·1	72·0	129·1	34·4	130·4	39·7	71·6	—	68·3	26·7	55·4	129·4	100·8	180·8	96·5	86·6	99·8
	Hottentots ...	—	71·9	126·0	32·3	133·6	38·2	70·6	70°·6	65·9	26·5	58·0	131·1	101·8	185·7	95·2§	84·8	99·6
	Teita	59·5	70·6	130·9	33·9	129·6	39·7	71·4	69°·2	68·5	27·6	58·3	130·8	98·9	183·9	96·1	85·6	101·5
Other Races	Bushmen ...	—	75·6	123·8	31·1	135·7	38·2	70·9	69°·8	62·1	25·6	58·2	126·9	106·9	179·3	94·1	81·4	95·2
	Tasmanians ...	—	74·2	131·0	31·05	136·0	39·3	71·3	73°·5	62·5	27·8	59·1	130·9	103·9	182·2	94·0	79·4	98·8
	Badari Egyptians	60·6	71·8	122·5	32·0	130·8	38·4	73·1	66°·4	67·1	24·9	53·0	132·9	98·3	182·3	91·1	83·3	99·3

* Some of the *a*'s for orbital breadths and indices included in these totals were found from the O_1 or *Lacr.* O_1 measurements in cases where O_1' was not available for both series.
† Some of the *a*'s for nasal indices included in this total were found for 100 *NB/NH* in cases where 100 *NB/NH'* was not available for both series.
‡ Mean for 18 skulls. § Mean for 19 skulls. All other means given in this table are for 20 or more skulls.

ments, so they can be of little real use. Comparisons are made in Table IX between the remaining characters. Of these there are four having more than half their available α's significant. Sufficiently reliable means of the occipital index can only be given for six of the series. It has been suggested* that this character may serve to differentiate negro types from almost all others in the world, since the West African means are as high—indicating a peculiarly small curvature of the median sagittal section of the occipital bone—as any others known. The means for the Tanganyika and Kenya Colony tribes that can now be given are of the same order as those for dynastic Egyptian and European series, so a high index is not a peculiarity of all negro races, though the character appears to make a sharp distinction between those of West and East Africa. Other measurements which promise to serve the same purpose are $100\,B/L$, O_2, $100\,H'/L$, and possibly O_1'. The West African types have the larger orbital heights and breadths, but none of the other absolute measurements with which we are dealing are peculiar to them. A single character can only suggest a linear arrangement of the types, which is obviously inadequate, and not even such an important one as the cephalic index is capable of providing an order which is very similar to, or as suggestive as, the best which might be derived from the coefficients of racial likeness. The Gaboon and Bushmen series have identical cephalic indices, but they are clearly distinguished by the almost equally important bizygomatic breadths and facial heights (O_2 and $G'H$). If two characters are considered together—say $100\,B/L$ and O_2—a rather more suggestive classification may be made, but another pair—such as J and $G'H$—will lead to an appreciably different result. In any case no two dimensional arrangement can possibly reveal the complex inter-relationships of the types. The method of the coefficient of racial likeness has obvious advantages over any which depends on the comparison of single measurements and, in the present case, it appears to have led to suggestive results which could not have been reached by any other means at present available.

It may be noticed that the Badari Egyptian series is distinguished from all others in Table IX by having the smallest J, B', and NB. It has also the smallest nasal angle, while the Galla and Somali have the only lower nasal index. The Tasmanians have the lowest orbital index and the highest nasal index, the largest nasal angle and the smallest orbital height, being thus distinguished from all the African series more by their shape than by their size. The Kaffir type is peculiar on account of the large size of its calvaria: its L, B', and LB are the largest in Table IX, while the J and B are larger than those for any of the other South or East African series. Judging from all the absolute measurements, the Hottentot and neighbouring Bushmen types are smaller than all the negro types.

Type Contours. The methods of drawing individual and constructing type contours used by earlier workers in the Biometric Laboratory were followed exactly in dealing

* Karl Pearson and A. G. Davin: "On the Biometric Constants of the Human Skull," *Biometrika*, Vol. XVI. 1924, pp. 328—363. The occipital indices for Angoni (♂ 60·7 (15), ♀ 62·7 (8)) and Bushmen (♂ 59·6 (9), ♀ 58·7 (13)) are given on p. 334 of the above paper, and these values may be compared with those given in our Table IX.

with the Teita and Naishi River cave series. In the case of the latter, there are only six skulls available for this purpose and no use has been made of the transverse and horizontal contours. The sagittal type is given in Fig. 9 as it shows the most salient characters of the series, though it is realised that the measurements are based on too small a sample to make it of any permanent value. The means from which the types were constructed are given in Tables X—XII and the Teita male and female sections are Figs. 3—8. A comparison between chords measured on the type contours and the corresponding means of chords taken directly from the skulls provides a satisfactory check on both. For the Teita series 24 comparisons of this kind are possible: all the differences are less than 2 mm. and there are only three greater than 1 mm. The larger discrepancies may have been partly caused by the fact that the two different kinds of means were not based on precisely the same number of measurements.

The transverse type contours are very nearly symmetrical, the differences between the right and left sides only exceeding 1 mm. for the parallels 9 and $A\frac{1}{4}$ in the case of the male type. The maximum difference between the sides in the female type is 0·6 mm., this section being apparently as symmetrical as any other that has yet been constructed. Neither possess any peculiar features. As for nearly all the previously published types, the maximum breadths lie between the fourth and fifth parallels. An index expressing the height of the maximum breadth as a percentage of the total height has been given by Woo*, the male indices ranging from 31·8 for Anglo-Saxons to 48·7 for ninth Dynasty Egyptians. The male Teita index is 39·1 and the female 36·1. A sexual difference of the same sign was found for all the other series. Crewdson-Benington has provided type contours for three negro series and Stoessiger for the Badari Egyptians. A comparison of other indicial measurements used by earlier workers fails to distinguish the Teita types, or to make any clear distinction between the negro and other races.

The horizontal sections are less symmetrical than the transverse. As far as their general appearance is concerned, and judging also by the most suggestive relative measurements, these contours are again found to be quite unexceptional in nearly all respects. One peculiarity may be noted—the temporal lines are relatively further back than on any previously constructed type. This feature has been measured by expressing the mean of the x co-ordinates of T_R and T_L as a percentage of FO. The male Teita index is 11·2 and the female 11·1. Ten other male types give a range from 9·6 to 11·0†. The horizontal Teita sections are very similar to those given for the Badari Egyptians, although there is no close correspondence between the transverse sections of the two series.

The section of the facial skeleton was not shown on the sagittal type contours for negro series given by Crewdson-Benington, and there are no others, except the Teita figures, available for a prognathous race. The Teita show no marked sub-nasal pro-

* *Biometrika*, Vol. XXII. 1930, p. 83. References to all the sources in which type contours have been previously published are also given on this page.

† *Annals of Eugenics*, Vol. IV. 1930, p. 150.

gnathism. The depression of the nasal bones can be seen clearly. In general appearance the calvarial section has no very unusual features, both the frontal and occipital bones being well developed. A number of indicial and angular measurements have been used to compare the most important features of these median sagittal contours*. These show that the Teita skulls possess unusually convex sections of the frontal bone. This character has been measured in two ways: the index expressing the maximum subtense from the base of the bone—the $N\beta$ chord—to the length of that chord is 24·7 for the Teita male contour and 25·1 for the female, while the male indices previously found range from 19·3 to 24·4 and all the female values exceed the corresponding male ones; the angle between the line joining that point of maximum subtense to the nasion and the $N\beta$ chord—the angle of frontal bone flatness—is 29°·5 for the present male contour and 29°·0 for the female, while the sexual differences for other series are all of the opposite size to this and the male values range from 22°·3 to 28°·6.

The very inadequately determined type contour for the skulls from the Naishi River cave shows one peculiar feature which can be easily appreciated from the skulls themselves. This is the remarkably retreating frontal bone. The angle of frontal bone flatness is not peculiar, but the angle $\beta N\gamma$, which measures the degree of rotation of that bone, is 43°·2, the lowest male value previously found being 43°·5. The physiognomic angle of flatness—the sum of the two angles considered above—is 69°·6 while the lowest male value previously found is 71°·0. There are obviously marked differences between the Naishi River and Teita series in these and other respects.

Appendix. Tables of Individual Measurements. Individual measurements of the Teita and Naishi River skulls are in the two following tables. Where no value is given it was not possible to determine a sufficiently accurate reading owing to the condition of the specimen, except that for the male Naishi River skulls the Glabella *U*'s and Lacrymal O_1 *R*'s were omitted in some cases in error. The capacities were found by tight packing with mustard seed and weighing in the manner described by Macdonell (*Biometrika*, Vol. XIII. p. 261). The weights of seed held by the *crâne étalon* were found on seven different occasions to be: 1053·5, 1063·5, 1066·5, 1060·0, 1063·5, 1054·0, and 1065·0 gms. and the mean of these is 1060·6. The water capacity of this skull is 1365 cm.^{3}, so the factor $\frac{1365}{1060{\cdot}6} = 1{\cdot}217$ was used to transform the weights (gms.) of seed packed into the individual skulls into capacities (cm.^{3}). The index letters denoting measurements in the tables and text of this paper are those habitually used by workers in the Biometric Laboratory. They are defined in *Biometrika*, Vol. XXI. pp. 82—83.

* *Annals of Eugenics*, Vol. IV. p. 150, 1930.

TABLE X.

Mean Measurements of Teita Transverse Contours.

Sex	Nos.	*MA*	$1R=1L$	$2R$	$2L$	$3R$	$3L$	$4R$	$4L$	$5R$	$5L$
♂	48	111·4	55·6	59·7	59·1	61·5	61·1*	63·5	63·1*	63·7	63·8
♀	51	106·0	53·6	57·1	56·6	59·3	58·9	61·3	61·2	61·4	61·6

Sex	$6R$	$6L$	$7R$	$7L$	$8R$	$8L$	$9R$	$9L$	$10R$	$10L$
♂	62·6	63·1	61·2	61·5	57·4	58·0	48·8*	50·0	34·7*	35·1
♀	60·8	61·0	59·3	59·6	55·7	55·9	48·0	48·1	33·9	34·1

Sex	$A\frac{1}{4}R$	$A\frac{1}{4}L$	ZR, R		ZR, L	
			y	x	y	x
♂	17·0	18·7	59·2	4·4	58·8	4·6
♀	16·4	17·0	56·7	2·9	56·4	3·5

* Number of cases = 47.

TABLE XI.

Mean Measurements of Teita Horizontal Contours.

Sex	Nos.	*FO*	$F\frac{1}{4}R$	$F\frac{1}{4}L$	$F\frac{1}{2}R$	$F\frac{1}{2}L$	$2R$	$2L$	$2\frac{1}{2}R$	$2\frac{1}{2}L$	$3R$	$3L$
♂	46	183·2	25·8	23·2	36·5†	35·4	47·6*	47·3*	47·8	47·9	49·0*	49·3
♀	49	174·4	22·9	21·0	32·7	31·4	45·5	44·6	45·7	46·0	47·1‡	47·3‡

Sex	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$	$7R$	$7L$	$8R$	$8L$
♂	53·6†	52·8*	60·5	59·5	64·1	63·3	64·7	64·3	62·3	62·1
♀	51·4‡	50·8‡	57·8‡	57·1	61·8‡	61·1	62·8	62·4	60·4	60·4

Sex	$9R$	$9L$	$10R$	$10L$	$O\frac{1}{4}R$	$O\frac{1}{4}L$	TR		TL	
							y	x	y	x
♂	55·4	55·6	41·7*	42·3*	23·8*	23·7	50·0*	20·3*	50·2	20·6
♀	53·6	53·7	40·9	41·3	23·1	23·2	47·5	18·7	47·9	19·9

* Number of cases = 45. † Number of cases = 44. ‡ Number of cases = 48.

TABLE XII.

Mean Measurements of Teita and Naishi River Sagittal Contours.

	Sex	Ordinates above $N\gamma$												
		0 at N	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$
Teita	♂	25·7 (46)	42·7 (46)	59·9 (46)	71·1 (46)	78·1 (46)	82·1 (46)	83·7 (46)	83·6 (46)	79·7 (46)	70·6 (46)	52·5 (46)	28·3 (46)	21·9 (46)
	♀	22·7 (49)	41·7 (49)	58·9 (49)	69·8 (49)	76·0 (49)	79·8 (49)	81·4 (49)	81·3 (49)	77·9 (49)	68·8 (49)	51·3 (49)	26·8 (49)	19·5 (49)
Naishi River	♂	13·3 (6)	27·9 (6)	52·3 (6)	66·9 (6)	74·7 (6)	78·9 (6)	81·6 (6)	82·6 (6)	79·0 (6)	70·4 (6)	51·6 (6)	24·1 (6)	19·0 (6)

	Sex	Ordinates below $N\gamma$								$N\gamma$	Vertex		β		Glabella		Occipital Pt.		λ		Sub-orbital Pt.	
		0 at N	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$		x from N	y	x from N	y	x from N	y	x from γ	y	x from γ	y	x from N	y
Teita	♂	61·2 (43)	58·1 (44)	51·5 (44)	50·2 (34)	43·5 (45)	33·3 (46)	25·3 (46)	17·8 (46)	181·0 (46)	96·7 (46)	84·4 (46)	73·2 (46)	82·3 (46)	1·9 (46)	9·2 (46)	0·5 (46)	2·8 (46)	5·7 (46)	30·3 (46)	8·2 (46)	28·2 (46)
	♀	58·2 (45)	55·2 (47)	48·9 (48)	47·2 (40)	42·2 (49)	30·9 (49)	23·4 (49)	16·4 (49)	173·9 (49)	94·0 (49)	81·9 (49)	71·2 (49)	80·0 (49)	0·8 (49)	10·3 (49)	0·4 (49)	2·9 (49)	5·4 (49)	28·8 (49)	8·4 (49)	26·4 (49)
Naishi River	♂	61·1 (5)	59·1 (5)	54·2 (5)	54·8 (4)	46·8 (6)	39·2 (6)	28·4 (6)	20·4 (6)	188·0 (6)	108·5 (6)	83·1 (6)	85·6 (6)	81·0 (6)	0·8 (6)	9·7 (6)	0·7 (6)	2·6 (6)	3·7 (6)	21·5 (6)	5·3 (6)	28·1 (6)

TABLE XII (*continued*).

	Sex	Auricular Pt.		Opisthion		Inion		Basion		Alveolar Pt.		Nose					Palate			
																	P'		*P*	
		x from γ	*y*	*x* from γ	*y*	*x* from γ	*y*	from *N*	from γ	from *N*	from Basion	*NL*	I	II	III	*LN*γ ∠	*x* from *N*	*y*	*x* from Alv.	*y*
Teita	♂	91·7 (46)	27·7 (46)	54·9 (37)	48·6 (37)	13·9 (46)	31·3 (46)	101·1 (38)	102·4 (38)	68·1 (32)	99·3 (27)	20·1 (19)	0·9 (45)	2·1 (43)	4·0 (38)	111°·8 (19)	45·6 (30)	48·6 (30)	37·2 (23)	16·7 (23)
	♀	88·4 (49)	25·3 (49)	53·0 (42)	47·1 (42)	12·9 (49)	28·2 (49)	96·3 (46)	99·3 (46)	63·7 (29)	94·2 (28)	19·8 (17)	0·6 (43)	1·7 (44)	3·3 (39)	107°·1 (17)	43·0 (36)	45·6 (36)	36·8 (26)	16·0 (26)
Naishi River	♂	105·6 (6)	27·7 (6)	66·6 (6)	54·9 (6)	20·7 (6)	40·8 (6)	96·2 (5)	115·8 (5)	67·8 (2)	90·9 (2)	—	1·3 (6)	3·1 (6)	6·8 (5)	—	42·8 (5)	52·3 (5)	36·8 (2)	13·5 (2)

	Sex	Frontal Max. Sub. to *Nβ*		Occipital Max. Sub. to λ*Op*		Max. Sub. to *N*λ		Max. Sub. to *GI*		Sp.		Prosthion		*N.S.*		Mid. Sub. from Bas.-Sp. Chord	
		x from *N*	*y*	*x* from λ	*y*	*x* from *N*	*y*	*x* from *G*	*y*	*x* from *N*	*y*	*x* from *N*	*y*	*x* from *N*	*y*	*x* from Basion	*y*
Teita	♂	48·1 (46)	27·2 (46)	47·4 (37)	27·5 (37)	82·5 (46)	69·2 (46)	103·1 (46)	99·5 (46)	69·4 (41)	33·4 (39)	9·2 (34)	64·2 (34)	5·0 (35)	47·6 (35)	12·5 (38)	0·3 (38)
	♀	48·2 (49)	26·9 (49)	42·6 (42)	25·2 (42)	79·4 (49)	67·3 (49)	98·6 (49)	94·6 (49)	66·1 (46)	32·0 (46)	8·5 (38)	60·6 (38)	3·6 (35)	44·6 (35)	11·5 (45)	0·3 (45)
Naishi River	♂	52·6 (6)	26·4 (6)	44·1 (6)	28·0 (6)	95·4 (6)	71·2 (6)	105·9 (6)	104·7 (6)	64·2 (6)	33·2 (6)	12·1 (3)	61·5 (3)	9·1 (5)	48·7 (5)	17·2 (5)	0·9 (5)

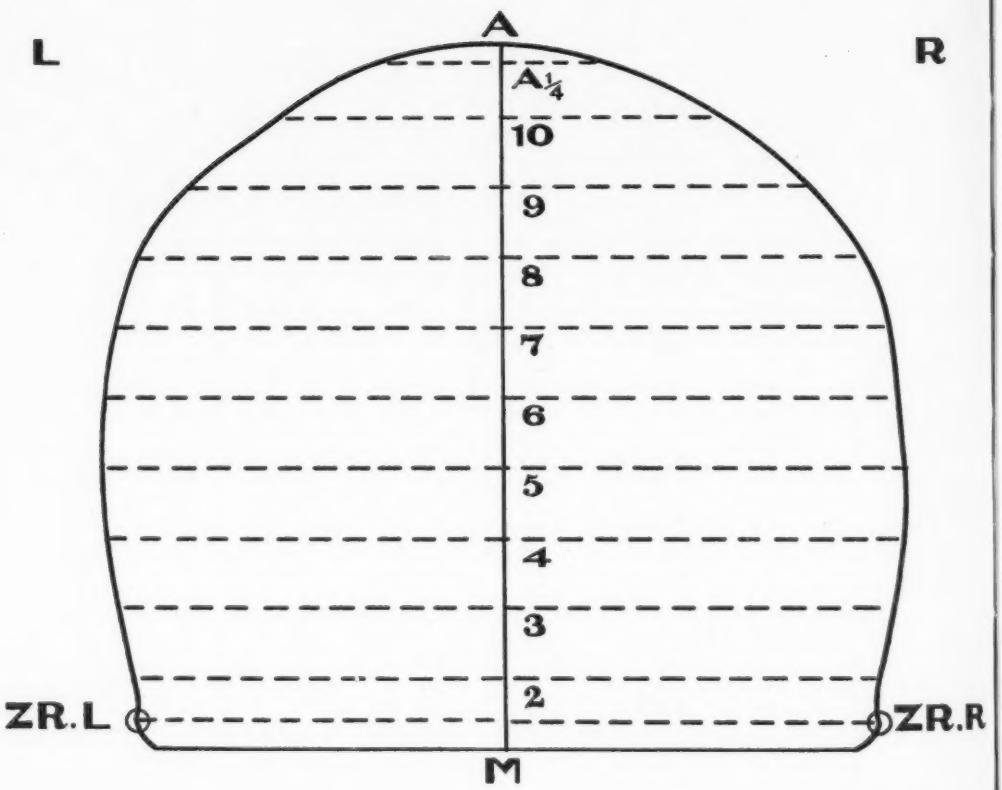


FIG. 3. Transverse Type Contour based on 48 ♂ Teita Skulls.

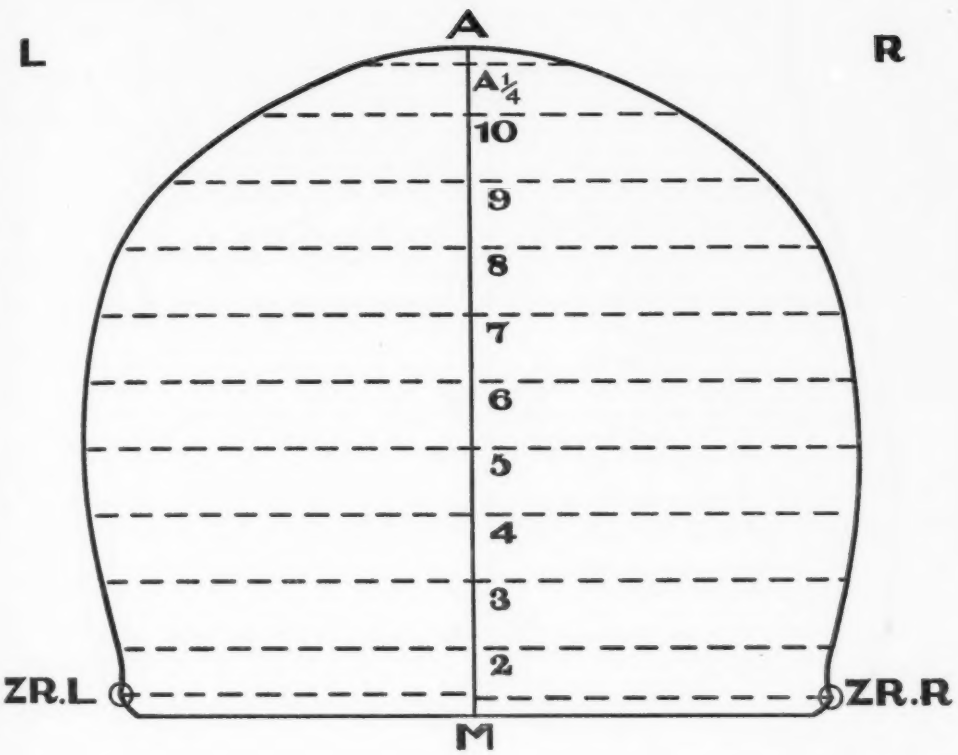


FIG. 4. Transverse Type Contour based on 51 ♀ Teita Skulls.

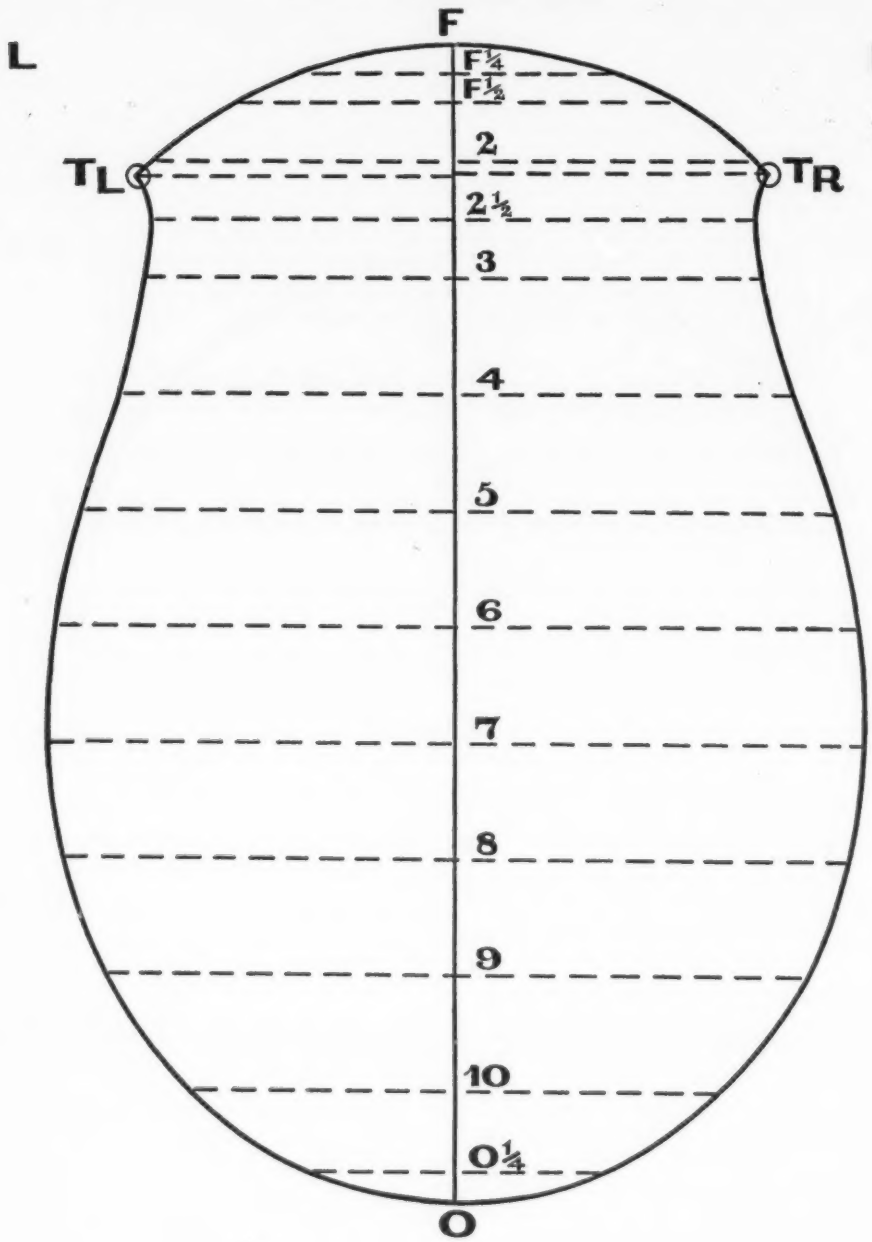


FIG. 5. Horizontal Type Contour based on 46 ♂ Teita Skulls.

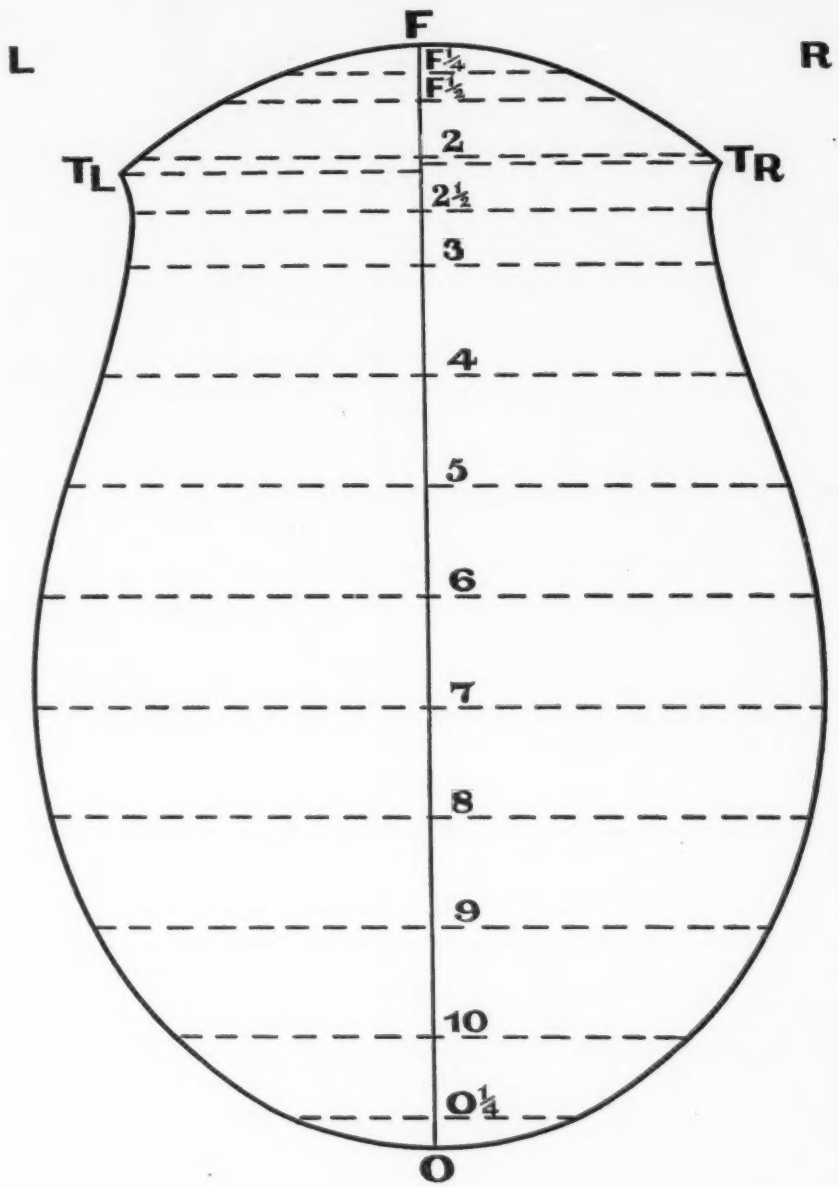


FIG. 6. Horizontal Type Contour based on 49 ♀ Teita Skulls.

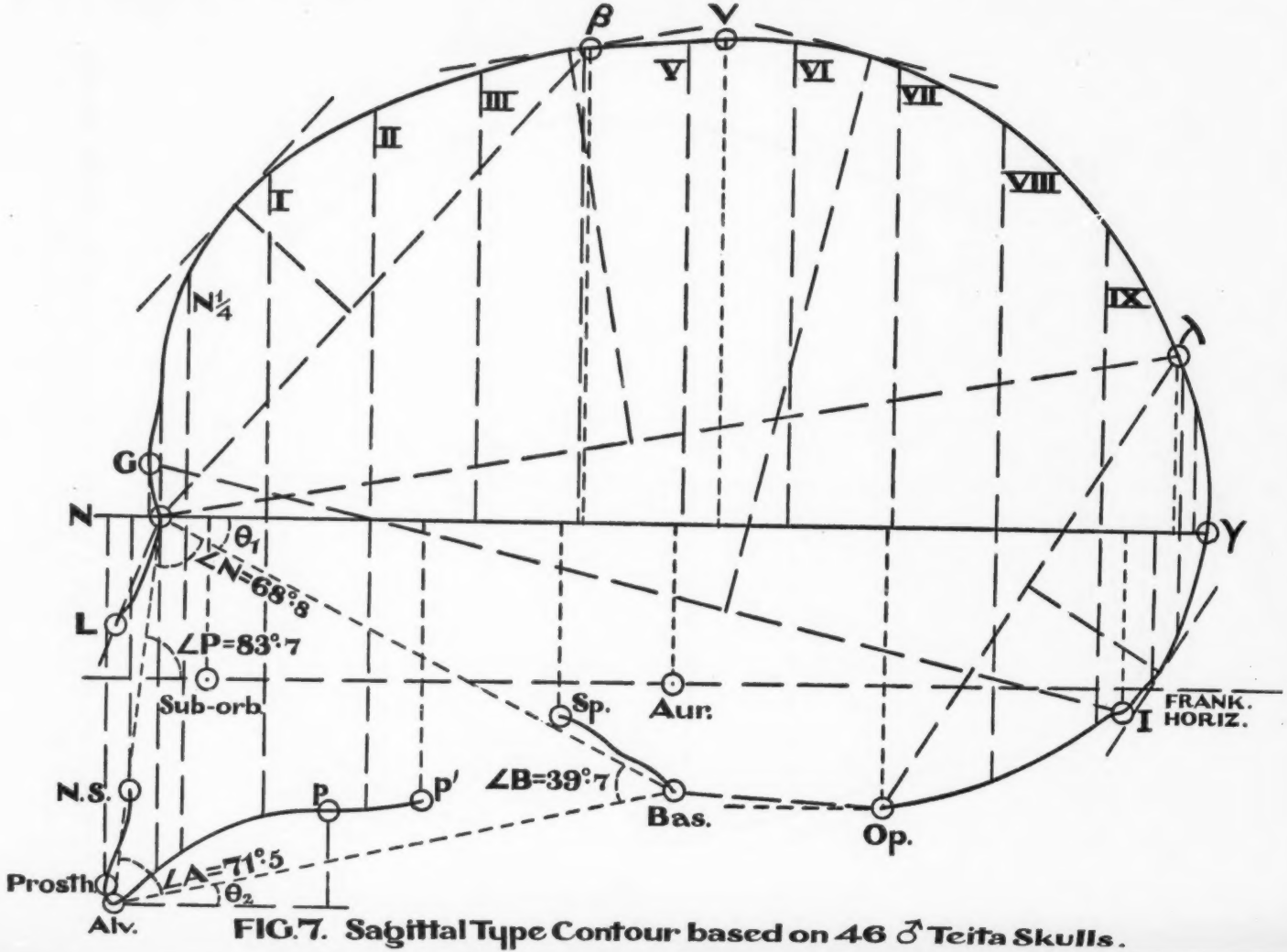


FIG. 7. Sagittal Type Contour based on 46 ♂ Teita Skulls.

Alv. FIG.7. Sagittal Type Contour based on 46 ♂ Teita Skulls.

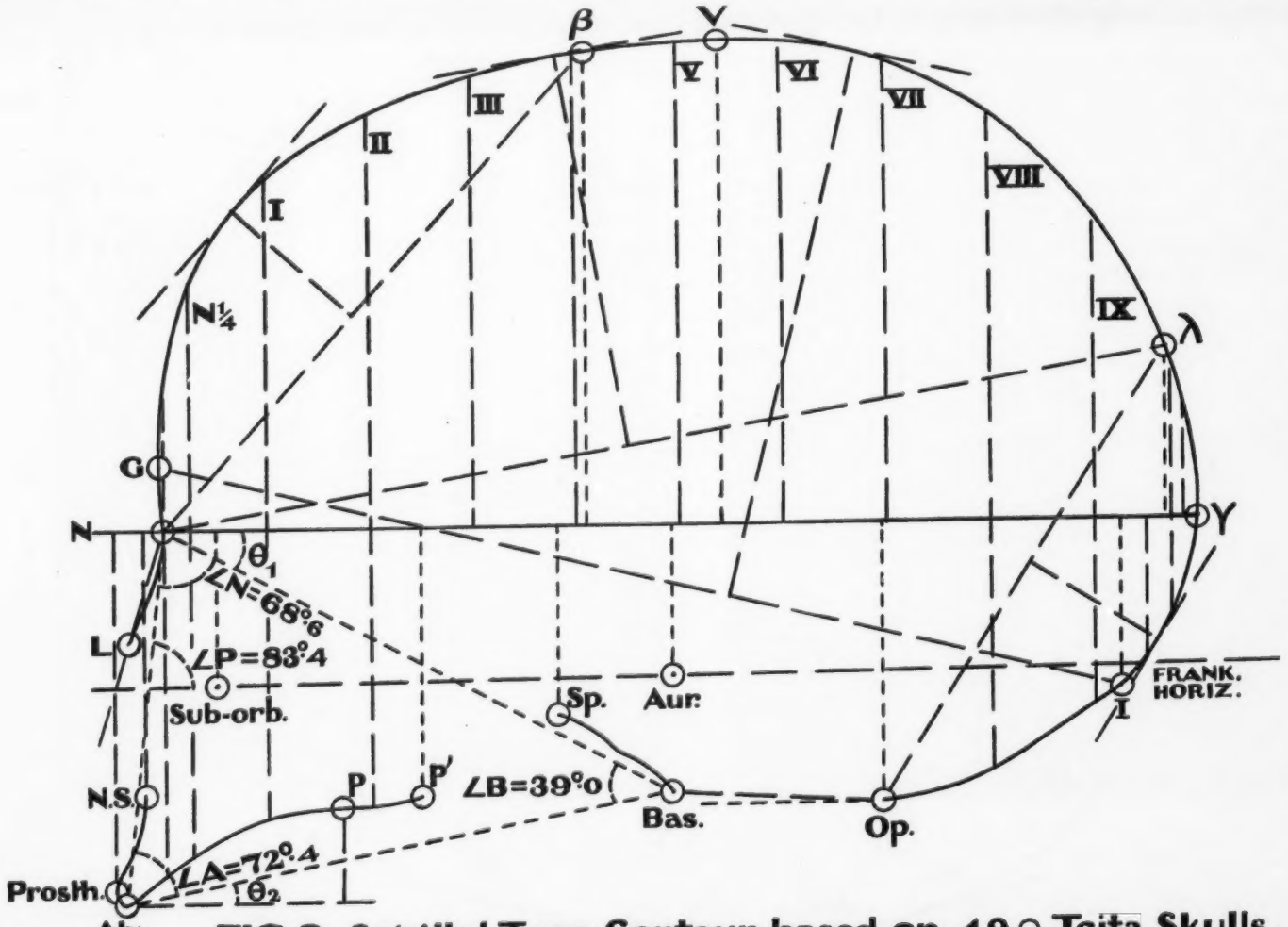


FIG.8. Sagittal Type Contour based on 49 ♀ Teita Skulls.

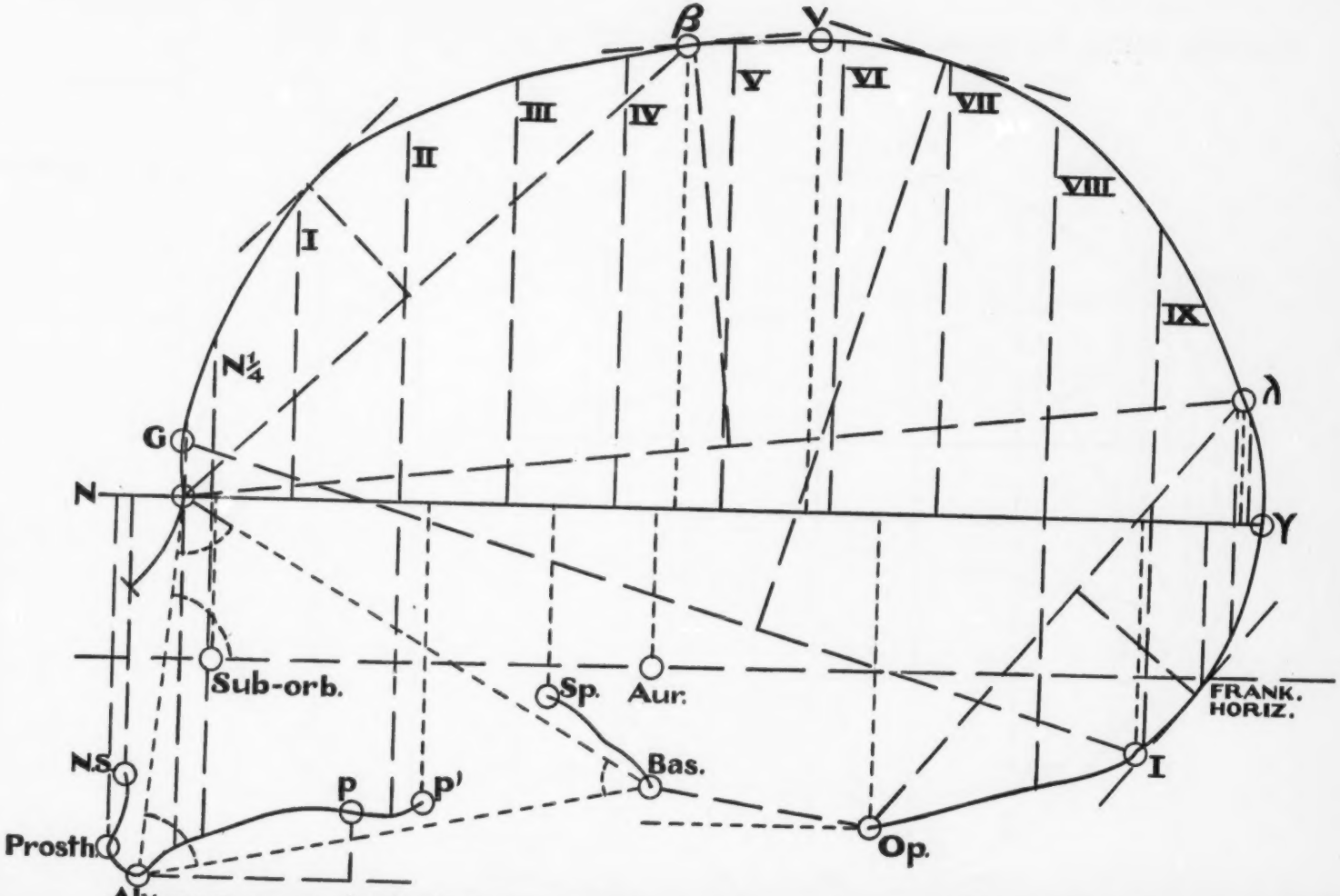
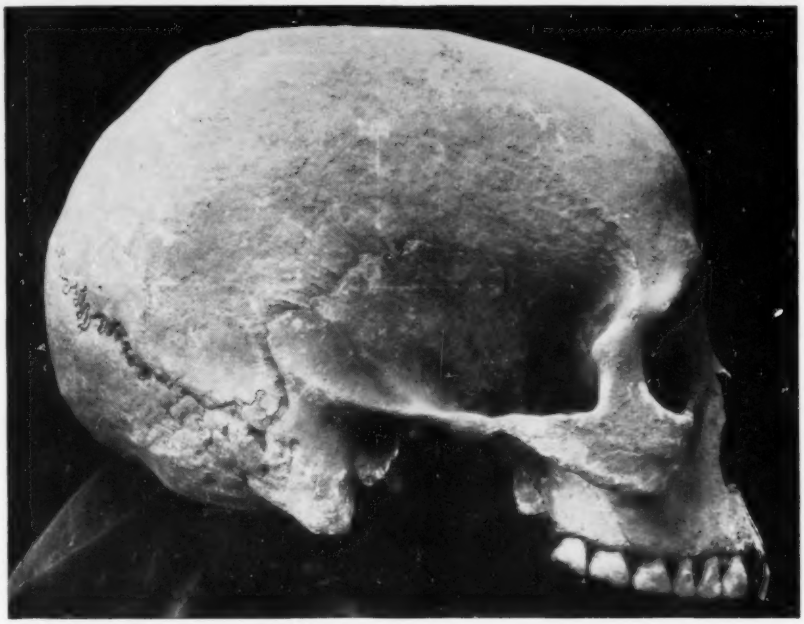


FIG.9. Sagittal Type Contour based on 6♂ Naishi River Skulls.



A. No. 16 ♂ (*ca.* 0·6 natural size). *Norma lateralis.*

B. No. 74 ♀ (*ca.* 0·7 natural size). *Norma lateralis*

Normal Teita Skulls

FIG. 9. Sagittal Type Contour based on 6 ♂ Naishi River Skulls.

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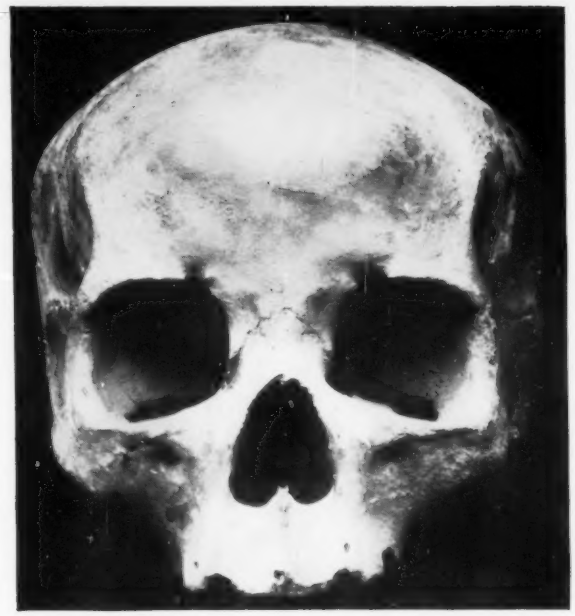
A. No. 16 ♂ (*ca.* 0·7 natural size). *Norma verticalis.*

B. No. 74 ♀ (*ca.* 0·7 natural size). *Norma verticalis.*

Normal Teita Skulls

UNIV. OF CH.



A. *Norma facialis* (*ca.* 0·6 natural size).

B. *Norma lateralis* (*ca.* 0·6 natural size).

Normal Male Skull from Naishi River Cave (E. 920, No. 7)

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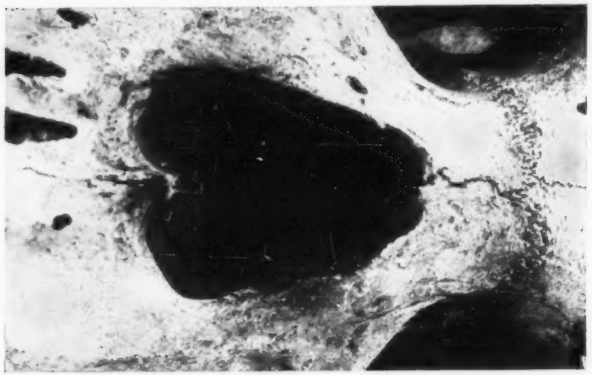
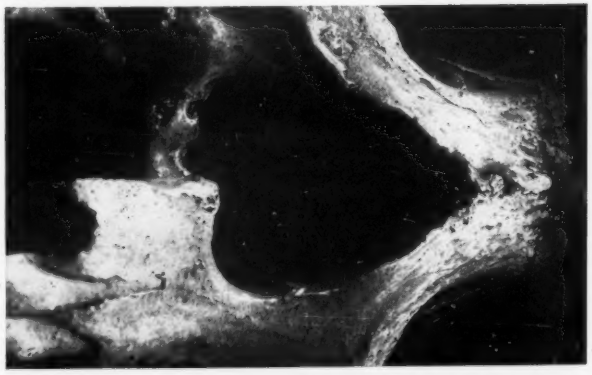
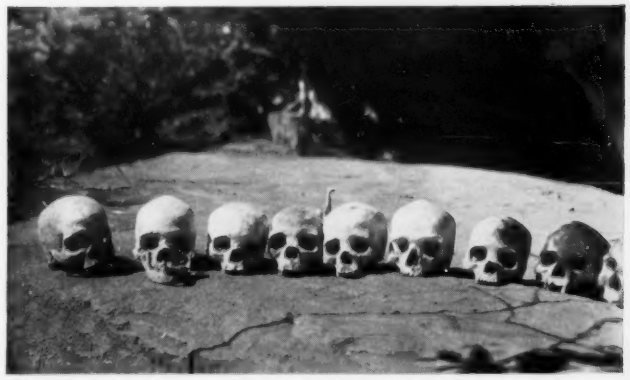


A. No. 121 ♀. Absence of nasal bones.

C. No. 25 ♂. Wormians in place of nasal bones.



B. No. 24 ♂. Wormians in place of nasal bones.



D. A Teita shrine.

Teita Skulls with anomalous Nasal Bridges (all ca. 1·4 natural size) and a Teita Shrine

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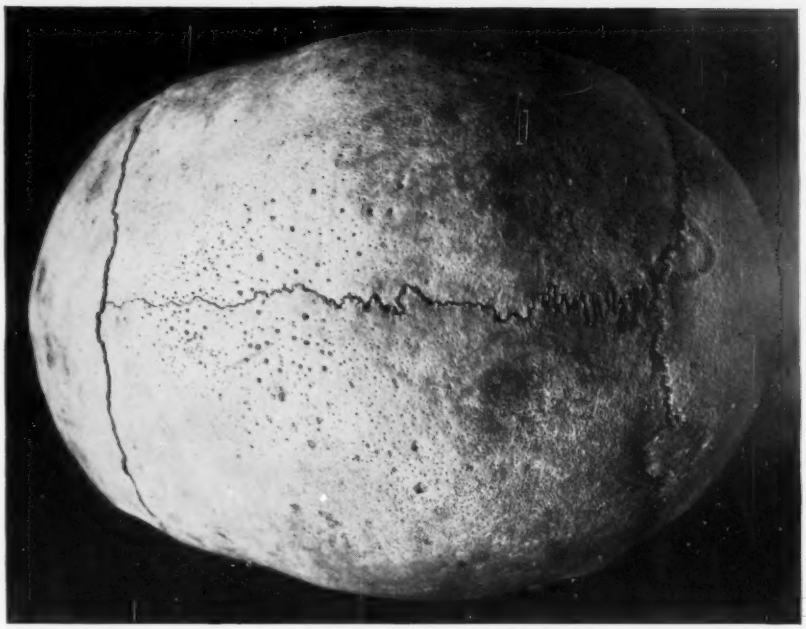

A. No. 11 ♂ (*ca.* 0·7 natural size). Pitting of parietal bones supposed due to periostitis.

B. No. 5 ♂ (*ca.* 1·6 natural size). Arthritic glenoid surface.

C. No. 12 ♂ (*ca.* 1·5 natural size). Arthritic glenoid surface.

Anomalous Teita Skulls

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	Serial No.	Cave No.	Sex	C	B′	B″	B	Biasterionic B	F	L	H′	H	OH	LB	
TEITA	KY 1	A 5	♂	1441	96·1	112 ?	131	102·6	191·5	189·5	135·5 ?	134·5	114	98·0	1
	2	A 8	,,	1429	106·0	114·5	131·5	110·1 ?	198	196·5	142 ?	140	120·5	108·1	1
	3	D 2	,,	—	96·5	—	133	104·4	194	193	—	—	118	—	1
	4	E 1	,,	—	92·1	113	132·5	105·7	180	179·5	119	119·5	106	99·4	1
	5	E 5	,,	1187	93·8	103·5	123·5	108·2	187	188·5	135·5	135·5	110	107·2	1
	6	E 7	,,	1418	97·3	112·5	129·5	106·4	191·5	193	139·5	140·5	119	110·8	1
	7	E 12	,,	—	95·9	113	138	114·0	192·5	191·5	—	—	116·5	—	1
	8	E 15	,,	—	94·1	110	128·5	96·7	187	186·5	130	131	115·5	105·8	1
	9	E 17	,,	—	97·1	109	126	103·1	190	189·5	137·5	—	—	106·2	1
	10	E 18	,,	1390	93·1	111·5	129·5	105·5	189	187·5	137·5 ?	137·5	115·5	103·2	1
	11	E 19	,,	—	91·6	111	132·5	110·7	189·5	190	129·5	132	118	101·4	1
	12	E 20	,,	1492	99·1	120	138	111·0	198·5	198·5	130·5	130·5	113·5	107·3	1
	13	E 28	,,	—	98·9	112	130·5	105·1	186·5	187	131	131·5	113	100·6	1
	14	E 30	,,	1149	97·0	106	128	104·6	178·5 ?	178	131	131·5	107	105·0 ?	
	15	E 34	,,	1138	89·9	100·5	119·5	104·3	180·5	182	129	128·5	108	103·0	1
	16	E 39	,,	1229	92·5	107·5	127	107·2	179·5	180·5	126·5	126·5	103·5	104·5	1
	17	E 40	,,	—	95·8	108	131	106·3	185·5	184·5	132 ?	130	111	110·5	1
	18	E 41	,,	—	02·6	—	134·5	105·0	186	185·5	—	—	114	—	1
	19	E 42	,,	1338	98·6	116	129	106·4	184	183·5	137	136	111	101·0	1
	20	E 43	,,	—	99·8	—	127	—	189	190	133	—	—	—	1
	21	E 44	,,	—	103·3	111·5	129·5	99·9	180	180	125·5	124·5	106	103·9	1
	22	E 47	,,	—	92·0	—	129·5 ?	—	182·5	186·5	—	—	—	—	1
	23	E 52	,,	—	80·0	—	—	—	—	—	—	—	—	—	
	24	E 55	,,	—	97·9	109·5	128	103·8 ?	173·5	173·5	128·5	129	112	94·0	1
	25	E 58	,,	—	94·1	—	132	—	181·5	181·5	—	—	113	—	1
	26	E 62	,,	—	97·8	111	127	110·9	186	186	131·5 ?	131·5	111	104·0 ?	1
	27	E 63	,,	—	95·9	—	—	—	—	—	—	—	—	—	
	28	F 3	,,	1291	—	112·5	135	104·6	177·5	183	124·5	124·5	111	101·5	1
	29	F 4	,,	1321	93·0	104	126	104·4	183·5	183	130·5	130	111·5	101·5	1
	30	F 6	,,	1176	95·8	112	131·5	100·5	177·5	178·5	123	123·5	110	100·0	1
	31	F 7	,,	1342	95·0	108	131·5	104·7	184	184	129	127·5	112	98·4	1
	32	F 8	,,	1389	97·3	112·5	128·5	106·0	189·5	188·5	133	132·5	111	103·6	1
	33	F 9	,,	1277	99·6	118·5	128	100·7	174	174	130·5	130·5	111·5	97·7	1
	34	G 1	,,	1342	106·3	118	131	103·7	187·5	187	133	131·5	114·5	99·8	1
	35	G 3	,,	1434	102·6	121·5	138	110·0	186·5	187·5	131·5	129	118	105·2	1
	36	G 5	,,	—	94·3	111·5	129·5	99·8	181·5	184·5	—	—	106·5	—	1
	37	G 6	,,	1220	97·8	106	120	101·5	179·5	179	129	128	110	105·7	1
	38	G 8	,,	—	104·0	113·5	128·5	100·5	183·5	182·5	132·5	132	112	96·5	1
	39	G 9*b*	,,	—	93·4	112	133	104·7	181	183	120	—	104·5	98·0	1
	40	J 2	,,	1219	98·2	111	124	99·7	175·5	174·5	124·5	123·5	108	92·3	1
	41	J 3	,,	1259	97·8	118 ?	128	103·0	181	180·5	125	125	110	100·8	1
	42	J 6	,,	1337	94·6	114·5 ?	129·5	101·6	185·5	184·5	131·5	131·5	115	100·2	1
	43	K 1	,,	1398	93·2	119	137	112·5	188	189·5	132	132	116	102·5	1
	44	K 3	,,	1577	91·0	121	139	103·1	187	185·5	135	134·5	117·5	94·5	1
	45	L 4	,,	—	94·5	110·5	130·5	103·7	189	188	—	—	112·5	—	1
	46	A 2	♂?	1250	89·0	110	132	104·6	174	176	130	130	110·5	102·0	1
	47	E 3	,,	—	98·7	106	127	100·1	183	185	—	—	109·5	—	1
	48	E 6	,,	1199	97·7	100	127·5	109·3	175·5	175	129 ?	129	110·5	97·0	1
	49	E 36	,,	—	97·3	106·5	121	99·7	179·5	179	—	—	—	—	1
	50	E 48	,,	—	99·1	119·5	133·5	104·7 ?	185	184	—	—	—	—	1
	51	E 51	,,	—	—	—	119·5	97·4	—	—	129·5	129	107·5	—	
	52	E 53	,,	—	96·0	109	131·5	104·5	164·5	164	130·5	129	106·5	95·2	1
	53	E 66	,,	—	86·9	—	—	—	—	—	—	—	—	—	
	54	G 2	,,	1207	91·1	104	123	99·6	177·5	175·5	132	131	108·5	97·5	1
	55	G 7	,,	1349	98·3	107	127·5	101·5	184	184·5	130	129·5	111·5	98·0	1
	56	L 2	,,	1328	100·5	112·5	129·5	105·1	182	181·5	130	129·5	111·5	101·0	1
	57	M 1	,,	1358	91·1	106·5	130	102·3	180	181	131·5	—	—	101·3	1
NAISHI RIVER	[E 920] 1		♂	1524	85·5	112·5	137	108·3	187	193	134·5	130	113	98·6	1
	,, 2		,,	1388	92·8	108·5	135	114·0 ?	187	191·5	133	133	109·5	100·8	1
	,, 3		,,	—	99·0 ?	110	138·5	119·2 ?	189	193	121·5 ?	—	—	98·5 ?	1
	,, 4		,,	—	96·8	115·5	139·5	104·4	188	190	—	—	—	—	1
	,, 5		,,	1422	97·5 ?	107	135·5	111·0	185·5	191·5	129·5	130·5	110	98·5	1
	,, 6		,,	—	95·0	116·5	—	110·8	191·5	194	—	—	111·5	—	1
	,, 7		,,	1421	98·5	116	141	107·1	184·5	186	127	127·5 ?	109	92·1 ?	1
	,, 8		,,	1435	95·4	120	141·5	100·5	178·5	179	130·5	129	107	93·0	1
	,, 9		,,	—	85·6 ?	109 ?	—	—	—	—	—	—	—	—	
	,, 10		,,	—	—	—	138·5	—	—	—	—	—	—	—	
	,, 11		,,	—	—	—	—	—	—	—	—	—	—	—	
	,, 12		,,	—	—	—	—	—	—	—	—	—	—	—	
	,, 13		,,	—	—	—	—	—	—	—	—	—	—	—	
	KY 126		,,	—	98·6	—	—	—	191	191·5	—	—	—	—	1

OH	LB	S_1'	S_2'	S_3'	S_1	S_2	S_3	S	U	Glabella U	Q'	Bregmatic Q'	G'H	GL	GB
4	98·0	106·7 ?	130·8 ?	100·2	123 ?	145 ?	123	390	515	521	304	306 ?	—	—	96·8 ?
0·5	108·1	120·3 ?	117·4 ?	97·2 ?	139·5 ?	125 ?	119 ?	384	535·5	544	315	315 ?	—	98·8	—
8	—	109·9	119·3 ?	—	128 ?	135·5 ?	—	—	530	533	313	308 ?	67·0	—	—
6	99·4	102·1	96·1	94·5	117·5	104	125	346·5	500	504	297	291	67·6	106·2	103·0
0	107·2	108·5	117·0	93·6	121·5	131·5	117	369	504	516	297	292	66·0	103·8	96·2 ?
9	110·8	110·3	112·2 ?	106·1 ?	125 ?	121·5 ?	132 ?	378	521	528	310·5	311	67·1	109·0	102·0
6·5	—	118·2	117·8	92·4	137·5	133·5	120·5	391·5	528	528	311	310	71·7	—	95·2
5·5	105·8	111·0	114·4	98·1	126	126·5	116	368	509	515	303	297	67·7	111·5	104·7
—	106·2	115·1	121·9	101·2	130·5	134	119	381	513	519	—	295	—	—	—
5·5	103·2	104·2 ?	131·9 ?	91·2	119 ?	151 ?	108	377	512	517	307	303	67·5	95·5	98·8
8	101·4	111·3	120·0	92·4	125	137	115	378	517·5	524	312	302	67·7	102·5	100·0
3·5	107·3	116·0	129·5	88·6	134	146·5 ?	106·5 ?	387	534	540	311	306	76·4	107·2	104·0
3	100·6	111·8	118·8	91·9	131	133·5	108·5	373·5	517	519	303	298	—	—	—
7	105·0 ?	—	104·0	90·4	—	115	109·5	—	493	504	292	288	—	101·4	—
8	103·0	108·1	110·3	89·2	122	122	107·5	351·5	489	495	285	281	63·9	100·9	92·6
3·5	104·5	105·2	109·3	88·1	121	122	108	350	500	511	285·5	285	75·0	105·5	96·0
1	110·5	113·3 ?	109·7 ?	90·5	130 ?	119·5 ?	108	357·5	513·5	518·5	300	299 ?	75·0	102·4	98·5
4	—	108·9	123·3	—	123	136 ?	—	—	509	513	309·5	308·5	68·2	—	—
1	101·0	110·0	120·1	92·9	130	133	113	377	512	507	296	297	—	—	90·3 ?
—	—	110·5	116·2	96·8	124	129	117	370	514	517	—	294	—	—	—
6	103·9	104·9	111·5	93·5	119	122	109·5	351	506	508·5	286	287	70·5	113·0	102·0
—	—	114·6	116·1	—	132	131·5	—	—	511 ?	513 ?	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	103·0 ?
2	94·0	106·1	108·1	—	124	122	—	—	495	495	301	295	56·8	96·0	89·9
3	—	103·3	124·0	—	121·5 ?	142 ?	—	—	505	507·5	310	308·5	—	—	92·5
1	104·0 ?	106·9	111·0	96·5	123	122	118·5	363	512	517	300	299	73·2	96·2 ?	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
1	101·5	110·5	105·8	94·3	125	117	117·5	359·5	498·5	—	305	302	74·0	98·7	99·5 ?
1·5	101·5	114·3	108·2	90·8	133	122	110·5	365·5	499	504 ?	298	297	—	—	95·1
0	100·0	108·9	105·8	88·2	123	118	108	348	496·5	—	303	299	—	—	94·5
2	98·4	117·0	113·4	89·1	137·5	124	107·5	369	508	509	302	301	66·4	94·0	103·9
1	103·6	110·1	107·8	108·0	131	116·5	135·5	383	521	525	301	300	63·1	97·2	—
1·5	97·7	110·0	111·8	88·1	127·5	123·5	102	352·5	493	494	302	302	73·4	97·3	108·4
4·5	99·8	115·2	116·4	94·0	132·5	128	113·5	374	523	524	309·5	308	—	—	—
8	105·2	113·9	110·5	100·5	133	120	126·5	380	524	529	319	319	76·8	107·3	112·8
6·5	—	104·0	114·1	89·5	124	128	108	359	506	507·5	296·5	298	—	—	—
0	105·7	107·7	113·6	84·5	126	124·5	102	352	494	498	294	293	69·0	102·5	101·8
2	96·5	107·1	116·6	96·2	126·5	130·5	112	369	513	515	300	298	70·4	94·8	95·4
4·5	98·0	109·0	112·5	—	124·5	126·5	—	—	508	508	289	287	68·5	105·4	100·5
8	92·3	106·8	109·3	90·6	127·5	120	112	359	489	492	291	294	66·7	86·8	90·0
0	100·8	110·2	105·2 ?	94·1 ?	127	116 ?	114 ?	357	504	505	301	298	—	—	93·4 ?
5	100·2	111·3	119·5	90·1	127	133	107	367	510	510	302	301	66·7	101·6	94·0
6	102·5	116·4	116·8	97·7	130·5	130	120	380	520	523	317	313	—	—	105·0
7·5	94·5	113·7	126·4	95·7	135·5	144	118	397	515	513	319	317	62·5	92·3	100·2
2·5	—	113·0	110·0	—	131	123	—	—	519	522	301	299	—	—	102·4
0·5	102·0	108·1	109·5	92·2	124	121·5	107	352	485	491	294·5	292·5	—	—	94·6
9·5	—	107·3	107·1	93·4 ?	123·5	120	109·5 ?	352 ?	502	506	296	293	65·3	—	98·2 ?
0·5	97·0	103·8	112·9	95·5	118	129	112·5	359·5	490	491	294	287	71·5	98·0	—
—	—	108·7	118·9	—	122	134	—	—	490 ?	494 ?	—	—	68·5	—	—
—	—	107·8	115·7	—	126	126	—	—	512	511	—	306	—	—	—
7·5	—	—	113·0	86·4	—	126·5	103	—	480 ?	—	288	285	—	—	—
6·5	95·2	104·9	105·9	81·0	118	117	97	332	477	478	290	289	—	—	94·2
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8·5	97·5	113·7	108·7	90·1	134·5	119	106·5	359·5	485	488	285	285	60·1	95·2	91·6
1·5	98·0	109·2	110·8	99·5	124·5	123	121	368	496·5	497	302	297	—	97·2 ?	98·2
1·5	101·0	113·6	111·1	—	130	123·5	—	—	503·5	—	298·5	296	66·8	93·7	87·6
—	101·3	114·3	103·9	99·6	127	112·5	121	361	499·5	503	—	292	—	94·5	—
3	98·6	129·8	105·3 ?	99·2 ?	146	115 ?	128 ?	389	525	—	302	306·5	70·4 ?	102·0 ?	102·5
9·5	100·8	114·2	116·9	101·7	122	134	122	379	513	—	294	291	—	—	—
—	98·5 ?	118·7	109·5	93·0	131·5	121	115·5	368	535	—	—	291·5	—	—	—
—	—	123·5	109·1	96·8	140	119	118	376·5	524·5	—	—	309·5	—	—	—
0	98·5	116·5	116·9	102·0	129·5	131·5	123	383	518·5	—	296·5	297	—	—	—
1·5	—	117·2 ?	114·4 ?	102·5 ?	135 ?	128 ?	125 ?	388	534·5	—	310 ?	310 ?	—	—	94·0
9	92·1 ?	119·6	117·7	93·2	133·5	132	109	374	516	—	301	303	—	—	97·0
7	93·0	117·6	111·1	97·7	131·5	124·5	114·5	369	501	—	299	300	66·2	90·6	89·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	117·0	—	—	130	—	—	—	—	—	—	—	—	—
—	—	—	—	103·7 ?	—	—	127·5 ?	—	—	—	—	—	—	—	—
—	—	—	—	93·5	—	—	121	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	118·9	126·3	99·5	137	142	118	397	—	—	—	—	—	—	—

TABLE XIII *Individual Measure*

GL	*GB*	*J*	*PH*	*NH, R*	*NH, L*	*NH'*	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	O_1R	O_1L	$O_1'R$	$O_1'L$	*Lacr.* O_1R	O_2R	O_2
—	96·8 ?	133·5	—	48·5	48·4	45·0	27·5	—	—	—	3·0	12·0	41·3	41·8	38·6	—	37·0	31·9	31·
98·8	—	—	—	51·0	50·3	48·0	29·0	—	—	—	—	—	—	44·9	—	—	—	32·0	33·
—	—	—	19·0 ?	50·1	50·7	49·6	—	—	—	—	1·8	8·0	—	45·6	—	—	—	—	—
106·2	103·0	—	—	49·8	49·4	44·9	27·5	—	—	—	2·5	9·7	43·0	42·9	—	—	—	33·0	33·
103·8	96·2 ?	132·5	—	48·8	48·8	—	27·0	10·5	25·0	36	2·3	11·0	44·2	44·7	41·4	41·5	39·2	36·0	33·
109·0	102·0	136·5	19·0	52·3	53·2	47·5	30·4	—	—	—	2·1	9·0	43·0	43·0	—	—	—	33·5	33·
—	95·2	129	23·0	50·0	50·4	48·8	26·3	—	—	—	0·9	6·9	43·2	43·0	39·2	—	37·8	34·0	33·
111·5	104·7	133	—	51·4	50·5	—	26·0	12·9	26·0	41·5	2·3	10·4	44·9	44·3	—	40·3 ?	—	32·2	32·
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
95·5	98·8	128·5	20·0	50·9	50·9	47·7	28·0	—	—	—	0·8	5·9	41·0	42·1	—	—	36·7	32·2	34·
102·5	100·0	—	24·6	47·8	48·0	44·0	26·0	9·4	23·0	31	1·7	7·1	43·9	43·1	40·1 ?	40·5 ?	38·8	31·7	31·
107·2	104·0	142·5	24·0	58·0	58·0	51·3	29·0	—	—	—	2·7	6·3	46·2	46·3	41·0 ?	41·8 ?	41·6 ?	36·9	36·
—	—	—	—	—	49·9	—	—	—	—	—	—	—	—	43·4	—	—	—	—	32·
101·4	—	—	—	—	—	—	31·2	—	—	—	—	—	44·5	—	41·0 ?	—	—	30·0	—
100·9	92·6	123·5	—	—	48·0	—	24·5	—	—	—	3·0	9·7	45·3	44·9	—	—	—	34·7	34·
105·5	96·0	133	23·0	54·0	52·9	52·0	30·0	10·9	24·8	34	2·4	10·0	43·9	41·7	40·9	39·1	38·5 ?	32·5	32·
102·4	98·5	131·5	19·3	56·0	54·4	55·0	26·0	11·7	24·6	37·5	3·0	14·0	42·6	43·0	39·8	40·3	39·5	37·0	35·
—	—	—	—	51·0	50·9	46·8	27·0	9·1	22·9	31	2·6	10·0	—	44·0	—	40·5	—	37·0	38·
—	90·3 ?	—	—	50·4	50·8	50·3	25·1	7·9	23·9	32	4·2	10·6	42·8	43·0	—	—	—	34·2	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
113·0	102·0	—	25·0	48·0	47·5	45·9	25·8	10·0	27·0	36	0·4	6·7	42·8	44·1	40·0	40·9	38·7 ?	30·2	29·
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35·
—	103·0 ?	—	—	53·6	52·5	48·0	31·2	—	—	—	—	—	43·5	44·0	—	—	—	31·4	31·
96·0	89·9	—	—	—	40·0	37·8	—	8·6	21·2	30	—	—	42·8	41·0	39·0	38·0	37·7	31·2	31·
—	92·5	—	—	48·0	48·8	—	25·0	—	—	—	—	—	42·0	42·5	38·2 ?	39·1 ?	—	34·0	34·
96·2 ?	—	132·5	21·0	53·8	53·7	52·8	25·5	11·8	26·0	37	—	—	44·0	43·0	40·8	38·8	38·9	34·0	34·
—	—	—	—	54·0	54·0	54·5	28·4	10·5	27·9	37·5	2·2	12·0	45·7	44·9	44·3	43·1	—	37·0	36·
98·7	99·5 ?	—	—	53·0	53·5	51·0	26·0 ?	9·8	24·3	34	0·2	2·8	—	—	—	—	—	—	36·
—	95·1	126 ?	—	52·0	51·1	—	29·4	—	—	—	—	10·5	44·3	42·5	—	—	—	34·5	34·
—	94·5	—	—	49·8	49·2	—	28·0	—	—	—	2·6	11·0	44·0	44·1	—	—	—	33·7	33·
94·0	103·9	130·5	19·8	49·0	47·8	47·0	27·0	—	—	—	2·7	11·5	42·1	42·5	—	—	—	—	33·
97·2	—	—	21·0	44·3	44·0	42·8	26·0	—	—	—	—	—	45·3	44·5	—	—	—	32·5	32·
97·3	108·4	133	25·2	50·0	49·0	—	27·5	—	—	—	1·4	6·0	45·7	45·8	41·0	—	39·3	34·4	34·
—	—	—	—	46·4	46·7	43·9	28·0	—	—	—	1·9	12·0	44·2	43·5	41·4	—	—	34·0	32·
107·3	112·8	134·5	30·0	48·0	48·0	46·6	27·5	—	—	—	1·8	10·0	45·3	43·8	—	—	—	36·0	36·
—	—	132	—	49·2	48·1	—	30·0	—	—	—	—	—	43·5	42·8	—	—	—	—	35·
102·5	101·8	125	—	48·5	47·5	46·0	26·9	—	—	—	0·8	6·0	49·0	47·2	42·0	42·2	—	37·2	37·
94·8	95·4	127	19·4	52·5	52·0	51·9	28·3	11·2	27·0	37	2·7	10·1	42·0 ?	43·5	39·4	38·9	—	32·8	32·
105·4	100·5	129	—	50·0	48·0	44·0	29·2	10·8	24·6	36	2·4	10·7	43·5	43·0	40·0	38·8	39·2	33·9	32·
86·8	90·0	126	21·3	43·0	46·9	45·8	28·3	13·4	28·0	42	3·9	12·6	41·5	41·9	36·1	36·2	36·0	35·9	36·
—	93·4 ?	131·5	—	51·0	51·0	—	28·3	—	—	—	—	—	43·1	45·1	39·9	—	—	35·5	36·
101·6	94·0	134	19·5	48·8	49·0	47·0	25·5	9·8	23·4	33	1·8	8·2	45·9	43·9	42·0	39·9	39·5	34·0	35·
—	105·0	134	—	51·5	50·1	50·8	27·7	—	—	—	0·7	7·0	45·0	44·0	—	—	—	36·0	36·
92·3	100·2	137·5	—	50·4	50·5	—	30·6	—	—	—	1·1	10·3	—	38·5	—	—	—	32·0	32·
—	102·4	—	—	49·0	50·3	46·0	29·3	—	—	—	2·4	9·9	42·9	44·0	38·3	—	—	37·2	36·
—	94·6	—	—	47·0	47·0	45·0	28·3	10·5	21·9	32·5	2·5	8·9	41·0	41·1	37·2	36·9	36·8	32·7	31·
—	98·2 ?	—	20·8	48·5	47·8	44·3	27·5	10·4	23·0	35	2·5	11·0	40·0	42·1	37·0	38·1	35·7	—	31·
98·0	—	—	—	53·2	53·3	—	—	11·1	23·0	34	2·0	10·0	43·9	—	38·9	—	37·1	33·0	—
—	—	—	—	52·5	53·0	49·2	28·2	—	—	—	—	—	44·5	44·2	—	—	—	35·5	35·
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31·8	—
—	94·2	126·5	—	45·0	44·6	40·5	28·0	8·6	22·5	30	2·0	10·5	41·9	42·0	39·0	40·0	36·9	31·3	32·
—	—	—	—	—	47·7	44·5 ?	—	—	—	—	1·6	5·9	—	44·9	—	—	—	—	34·
95·2	91·6	123·5	18·5	43·8	44·1	42·7	23·8	8·8	23·0	31·5	2·2	9·5	42·0	40·2	37·3	36·7	—	—	33·
97·2 ?	98·2	—	—	46·7	46·1	45·0	27·5	8·5	23·0	30·5	—	—	42·9	42·2	38·7	39·0	37·2 ?	34·0	33·
93·7	87·6	—	18·0	52·4	52·0	46·8	26·9	10·4	23·2	32	3·6	10·8	42·0	42·8	39·2	39·0	—	38·7	37·
94·5	—	—	—	49·5	49·8	47·2	29·2	—	—	—	3·3	10·9	—	—	—	—	—	—	—
102·0 ?	102·5	—	18·0 ?	54·1	54·2	50·8	29·5	11·0	25·2	36	1·9	7·0	49·3	49·0	45·0	44·5	—	39·0	38·
—	—	134	—	52·8	52·2	51·0	28·0	11·0	24·0	35	3·4	10·1	44·8	—	40·0	—	—	36·3	—
—	—	—	—	—	—	—	—	—	—	—	3·8	10·7	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	3·8	11·5	—	45·0	—	41·1	—	—	34·
—	94·0	129	—	53·0	52·1	51·0	25·6	10·3	18·3	32	1·5	7·7	45·0	43·5	40·0	40·2	—	35·0	33·
—	97·0	120·5 ?	—	51·8	50·8	48·0	26·8	—	—	—	1·8	9·2	44·5	44·3	43·0	—	—	37·3	37·
90·6	89·0	122	19·1	48·7	49·0	49·0	22·5	9·9	21·3	30·5	1·4	8·8	44·0	45·0	41·0	40·5	—	37·7	37·
—	—	—	—	—	—	—	—	—	—	—	3·6	10·8	—	42·9	—	—	—	—	36·
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	3·4	10·9	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

al Measurements of Male Skulls from Kenya Colony.

cr. $_1R$	O_2R	O_2L	G_1'	G_1	G_2	EH	fml	fmb	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{H}{L}$	100 $\frac{B}{H'}$	100 $\frac{B}{H}$	100 $\frac{(B-H')}{L}$	$Oc.\ I.$	100 $\frac{fmb}{fml}$
7·0	31·9	31·8	—	—	—	—	35·4	29·0	69·1	71·5 ?	71·0	96·7 ?	97·4	−2·4 ?	58·2	81·9
—	32·0	33·7	47·2	50·9	—	—	41·6	—	66·9	72·3 ?	71·2	92·6 ?	93·9	−5·3 ?	58·4 ?	—
	—	—	—	—	—	—	—	—	68·9	—	—	—	—	—	—	—
—	33·0	33·1	—	—	43·0	12·3	32·8	28·7	73·8	66·3	66·5	111·3	110·9	7·5	54·7	87·5
9·2	36·0	33·9	49·9	53·9	35·8	8·0	34·2	30·1	65·5	71·9	71·9	91·1	91·1	−6·4	57·1	88·0
—	33·5	33·0	49·9 ?	—	33·9	10·7	35·8	29·3	67·1	72·3	72·8	92·8	92·2	−5·2	57·3 ?	81·8
7·8	34·0	33·8	50·0	53·3	38·0	14·7	—	29·3	72·1	—	—	—	—	—	55·1	—
—	32·2	32·9	50·3	53·9	40·8	9·0	38·4	29·8	68·9	69·7	70·2	98·8	98·1	−0·8	61·5	77·6
—	—	—	—	—	—	—	35·7	28·8	66·5	72·6	—	91·6	—	−6·1	62·1	80·7
6·7	32·2	34·0	46·0	50·2	35·9	13·9	37·5	30·0	69·1	73·3 ?	73·3	94·2 ?	94·2	−4·3 ?	61·3	80·0
8·8	31·7	31·5	47·9	—	40·4	12·2	35·5	27·7	69·7	68·2	69·5	102·3	100·4	1·6	57·3	78·0
1·6 ?	36·9	36·0	50·8	57·1	40·8	13·4	39·0	28·9	69·5	65·7	65·7	105·7	105·7	3·8	59·9 ?	74·1
—	—	32·0	—	—	—	—	35·0	28·3	69·8	70·1	70·3	99·6	99·2	−0·3	61·6	80·9
—	30·0	—	—	—	39·4	8·2	38·0	28·0	71·9	73·6	73·9	97·7	97·3	−1·7	59·2	73·7
—	34·7	34·9	46·9	52·5	39·1	10·0	36·0	29·0	65·7	70·9	70·6	92·6	93·0	−5·2	59·6	80·6
8·5 ?	32·5	32·0	51·0	55·5	42·0	11·5	36·8	28·0	70·4	70·1	70·1	100·4	100·4	0·3	58·3	76·1
9·5	37·0	35·5	45·9	49·0	—	—	36·5	29·7	71·0	71·5 ?	70·5	99·2 ?	100·8	−0·5 ?	60·6	81·4
—	37·0	38·0	—	—	—	—	—	—	72·5	—	—	—	—	—	—	—
—	34·2	—	—	—	—	—	35·0	30·2	70·3	74·7	74·1	94·2	94·9	−4·4	58·9	86·3
—	—	—	—	—	—	—	38·8	31·5	66·8	70·0	—	95·5	—	−3·2	59·4	81·2
8·7 ?	30·2	29·0	54·9	58·0	41·9	13·0	34·1	28·0	71·9	69·7	69·2	103·2	104·0	2·2	62·6	82·1
—	—	35·5	—	—	—	—	—	—	69·4 ?	—	—	—	—	—	—	—
—	31·4	31·0	41·8 ?	—	—	—	—	—	—	—	—	—	—	—	—	—
7·7	31·2	31·4	43·8	48·4	39·0	9·1	—	—	73·8	74·1	74·4	99·6	99·2	−0·3	—	—
—	34·0	34·8	—	—	—	—	—	—	72·7	—	—	—	—	—	—	—
8·9	34·0	34·0	44·5	49·0	37·0	13·0	40·0	—	68·3	70·7 ?	70·7	96·6 ?	96·6	−2·4 ?	58·2	—
—	37·0	36·7	—	—	43·6	—	—	—	—	—	—	—	—	—	—	—
—	—	36·0	50·0	54·1	39·4	13·8	33·5	29·0	73·8	68·0	68·0	108·4	108·4	5·7	57·2	86·6
—	34·5	34·9	40·8 ?	45·1 ?	—	—	38·0	31·0	68·9	71·3	71·0	96·6	96·9	−2·5	58·8	81·6
—	33·7	33·5	44·0	50·0	38·0	11·1	40·0	31·2	73·7	68·9	69·2	106·9	106·5	4·8	58·4	78·0
—	—	33·4	43·4	—	—	—	37·2	29·5	71·5	70·1	69·3	101·9	103·1	1·4	59·5	79·3
—	32·5	32·0	49·9	—	—	—	34·0	30·0	68·2	70·6	70·3	96·6	97·0	−2·4	56·8	88·2
9·3	34·4	34·5	49·9	54·1	42·0	15·9	36·8	—	73·6	75·0	75·0	98·1	98·1	−1·4	64·0	—
—	34·0	32·6	44·8	48·0	—	—	39·2	29·9	70·1	71·1	70·3	98·5	99·6	−1·1	59·5	76·3
—	36·0	36·0	51·5	57·2	—	—	33·0	29·0	73·6	70·1	68·8	104·9	107·0	3·5	56·7	87·9
—	—	35·0	—	—	—	—	—	—	70·2	—	—	—	—	—	59·5	—
—	37·2	37·8	47·7	55·0	39·0	12·0	33·7	26·8	67·0	72·1	71·5	93·0	93·8	−5·0	59·5	79·5
—	32·8	32·0	—	—	38·3	—	37·9	32·0	70·4	72·6	72·3	97·0	97·3	−2·2	63·3	84·4
9·2	33·9	32·2	—	—	—	—	—	—	72·7	65·6	—	110·8	—	7·1	—	—
6·0	35·9	36·0	43·5	48·0	—	—	35·0	27·1	71·1	71·3	70·8	99·6	100·4	−0·3	57·7	77·4
—	35·5	36·0	43·2	47·8	36·9	14·2	38·0	31·6	70·9	69·3	69·3	102·4	102·4	1·7	59·2 ?	83·2
9·5	34·0	35·9	—	—	40·0	13·8	42·8	35·0	70·2	71·3	71·3	98·5	98·5	−1·1	61·0	81·8
—	36·0	36·5	—	—	—	—	37·0	31·0	72·3	69·7	69·7	103·8	103·8	2·6	58·2	83·8
—	32·0	32·2	45·9	51·5	38·3	14·9	35·7	28·1	74·9	72·8	72·5	103·0	103·3	2·2	57·9	78·7
—	37·2	36·8	42·5	46·0 ?	—	—	—	—	69·4	—	—	—	—	—	—	—
6·8	32·7	31·1	—	—	—	—	34·9	29·0	75·0	73·9	73·9	101·5	101·5	1·1	63·6	83·1
5·7	—	31·2	—	—	40·4	11·9	—	27·0	68·6	—	—	—	—	—	62·5 ?	—
7·1	33·0	—	—	—	38·3	10·6	34·2	29·3	72·9	73·7 ?	73·7	98·8 ?	98·8	−0·9 ?	61·9	85·7
—	35·5	35·0	—	—	—	—	—	—	67·6	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	72·6	—	—	—	—	—	—	—
—	31·8	—	—	—	—	—	32·0	25·9	—	—	—	92·3	92·6	—	60·6	80·9
6·9	31·3	32·0	—	—	—	—	34·8	31·8	80·2	79·5	78·7	100·8	101·9	0·6	60·2	91·4
—	—	34·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	33·8	46·9	51·3	—	—	34·3	30·5	70·1	75·2	74·6	93·2	93·9	−5·1	61·5	88·9
7·2 ?	34·0	33·5	49·0	53·5	—	—	33·5	29·8	69·1	70·5	70·2	98·1	98·5	−1·4	58·9	74·5
—	38·7	37·5	40·5	42·5	—	—	40·0	30·0	71·3	71·6	71·3	99·6	100·0	−0·3	—	75·0
—	—	—	—	—	—	—	34·4	31·0	71·8	72·7	—	98·9	—	−0·8	59·0	90·1
—	39·0	38·7	53·5	58·0	41·1	15·1	38·9	29·0	71·0	69·7	67·4	101·9	105·4	1·3	55·5 ?	74·6
—	36·3	—	46·3	53·2	37·9	14·6	38·0	31·0 ?	70·5	69·5	69·5	101·5	101·5	1·0	60·0	81·6 ?
—	—	—	—	—	—	—	40·0	31·0	71·8	63·0 ?	—	114·0 ?	—	8·8 ?	57·4	77·5
—	—	—	—	—	—	—	—	—	73·4	—	—	—	—	—	58·7	—
—	—	34·0	—	—	—	—	36·5	30·0	70·8	67·6	68·5	104·6	103·8	3·1	59·6	82·2
—	35·0	33·5	—	—	—	—	—	28·9	—	—	—	—	—	—	58·6 ?	—
—	37·3	37·3	44·8	50·0	—	—	44·8	31·2	75·8	68·3	68·5 ?	111·0	110·6 ?	7·5	62·7	69·6
—	37·7	37·3	41·6	45·0	34·3	9·3	37·2	32·0	79·1	72·9	72·1	108·4	109·7	6·1	62·5	86·0
—	—	36·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	58·1 ?	—
—	—	—	—	—	—	—	41·0	—	—	—	—	—	—	—	55·4	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	61·1	—

Oc. I.	100 $\frac{fmb}{fml}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH}, R$	100 $\frac{NB}{NH}, L$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1}, R$	100 $\frac{O_2}{O_1}, L$	100 $\frac{O_2}{O_1'}, R$	100 $\frac{O_2}{Lacr.\ O_1}, R$	100 $\frac{G_2}{G_1}$	100 $\frac{G_2}{G_1'}$	100 $\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alveolar* $P\angle$	*Prosthion* $P\angle$
58·2	81·9	—	61·1	56·7	56·8	—	25·0	77·2	76·1	82·6	86·2	—	—	—	—	—	—	—	—
58·4 ?	—	—	60·4	56·9	57·7	—	—	—	75·1	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	22·5	—	—	—	—	—	—	—	—	—	—	84°	82°
54·7	87·5	65·6	61·2	55·2	55·7	—	25·8	76·7	77·2	—	—	—	—	28·6	76°·4	65°·5	38°·1	78°	75°·5
57·1	88·0	68·6 ?	—	55·3	55·3	42·0	20·9	81·4	75·8	87·0	91·8	66·4	71·7	22·3	68°·8	75°·0	36°·1	82°·5	81°
57·3 ?	81·8	65·8	64·0	58·1	57·1	—	23·3	77·9	76·7	—	—	—	67·9 ?	31·6	70°·7	73°·8	35°·5	—	—
55·1	—	75·3	53·9	52·6	52·2	—	13·0	78·7	78·6	86·7	89·9	71·3	76·0	38·7	—	—	—	84°	82°
61·5	77·6	64·7	—	50·6	51·5	49·6	22·1	71·7	74·3	—	—	75·7	81·1	22·1	76°·5	67°·5	36°·0	79° ?	—
62·1	80·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
61·3	80·0	68·3	58·7	55·0	55·0	—	13·6	78·5	80·8	—	87·7	71·5	78·0	38·7	63°·8	76°·7	39°·5	87°	84° ?
57·3	78·0	67·7	59·1	54·4	54·2	40·9	23·9	72·2	73·1	79·1 ?	81·7	—	84·3	30·2	71°·4	69°·9	38°·7	86°	84°
59·9 ?	74·1	73·5	56·5	50·0	50·0	—	42·9	79·9	77·8	90·0	88·7	71·5	80·3	32·8	68°·9	69°·4	41°·7	85°	82°·5
61·6	80·9	—	—	—	—	—	—	—	73·7	—	—	—	—	—	—	—	—	—	—
59·2	73·7	—	—	—	—	—	—	67·4	—	73·2 ?	—	—	—	20·8	—	—	—	—	—
59·6	80·6	69·0	—	—	51·0	—	30·9	76·6	77·2	—	—	74·5	83·4	25·6	69°·0	74°·8	36°·2	81°·5 ?	80°
58·3	76·1	78·1	57·7	55·6	56·7	44·0	24·0	74·0	76·7	79·5	84·4 ?	75·7	82·4	27·4	69°·8	68°·4	41°·8	79°	76°·5
60·6	81·4	76·1	47·3	46·4	47·8	47·6	21·4	86·9	82·6	93·0	93·7	—	—	—	63°·5	75°·5	41°·0	90°·5	89°·5
—	—	—	57·7	52·9	53·0	39·7	26·0	—	86·4	—	—	—	—	—	—	—	—	86°	83°
58·9	86·3	—	49·9	49·8	49·4	33·1	39·6	79·9	—	—	—	—	—	—	—	—	—	—	—
59·4	81·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
62·6	82·1	69·1	56·2	53·8	54·3	37·0	6·0	70·6	65·8	75·5	78·0 ?	72·2	76·3	31·0	78°·0	64°·3	37°·7	76°·5	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	65·0	58·2	59·4	—	—	72·2	70·5	—	—	—	—	—	—	—	—	—	—
—	—	63·2	—	—	—	40·6	—	72·9	76·6	80·0	82·8	80·6	89·0	23·3	74°·5	70°·8	34°·7	79°	77°
—	—	—	—	52·1	51·2	—	—	81·0	81·9	89·0 ?	—	—	—	—	—	—	—	—	—
58·2	—	—	48·3	47·4	47·5	45·4	—	77·3	79·1	83·3	87·4	75·5	83·1	35·1	62°·8 ?	74°·5 ?	42°·7 ?	87°·5	86°·5
—	—	—	52·1	52·6	52·6	37·6	18·3	81·0	81·7	83·5	—	—	—	—	—	—	—	—	—
57·2	86·6	74·4 ?	51·0	49·1	48·6	40·3	7·1	—	—	—	—	72·8	78·8	35·0	66°·0	70°·6	43°·4	86°·5	83°
58·8	81·6	—	—	56·5	57·5	—	—	77·9	82·1	—	—	—	—	—	—	—	—	—	—
58·4	78·0	—	—	56·2	56·9	—	23·6	76·6	76·0	—	—	76·0	86·4	29·2	—	—	—	—	—
59·5	79·3	63·9	57·4	55·1	56·5	—	23·5	—	78·6	—	—	—	—	—	66°·3	73°·5	40°·2	89°·5	87°·5
56·8	88·2	—	60·8	58·7	59·1	—	—	71·7	71·9	—	—	—	—	—	66°·0	77°·5	36°·5	83°	79°·5
64·0	—	67·7	—	55·0	56·1	—	23·3	75·3	75·3	83·9	87·5	77·6	84·2	37·9	67°·6	68°·0	44°·4	80°	79°
59·5	76·3	—	63·8	60·3	60·0	—	15·8	76·9	74·9	82·1	—	—	—	—	—	—	—	—	84°·5
56·7	87·9	68·1	59·0	57·3	57·3	—	18·0	79·5	82·2	—	—	—	—	—	70°·0	67°·7	42°·3	85°	82°·5
59·5	—	—	—	61·0	62·4	—	—	—	81·8	—	—	—	—	—	—	—	—	—	—
59·5	79·5	67·8	58·5	55·5	56·6	—	13·3	75·9	80·1	88·6	—	70·9	81·8	30·8	68°·2	73°·1	38°·7	82°·5	—
63·3	84·4	73·8	54·5	53·9	54·4	41·5	26·7	78·1 ?	73·6	83·2	—	—	—	—	67°·5	69°·0	43°·5	80°·5	78°
—	—	68·2	66·4	58·4	60·8	43·9	22·4	77·9	74·9	84·8	86·5	—	—	—	76°·2	64°·7	39°·1	78°	76°·5
57·7	77·4	74·1	57·9	55·2	56·5	47·9	31·0	86·5	85·9	99·4	99·7	—	—	—	63°·7	72°·9	43°·4	88°	84°
59·2 ?	83·2	—	—	55·5	55·5	—	—	82·4	79·8	89·0	—	77·2	85·4	38·5	—	—	—	—	—
61·0	81·8	71·0	54·3	52·3	52·0	41·9	22·0	74·1	81·8	81·0	86·1	—	—	34·5	71°·7	69°·8	38°·5	80°	78°
58·2	83·8	—	54·5	53·8	55·3	—	10·0	80·0	83·0	—	—	—	—	—	—	—	—	—	—
57·9	78·7	62·4	—	60·7	60·6	—	10·7	—	83·6	—	—	74·4	83·4	38·9	68°·6	72°·4	39°·0	86°	83°
—	—	—	63·7	59·8	58·3	—	24·2	86·7	83·6	97·1	—	—	—	—	—	—	—	—	—
63·6	83·1	—	62·9	60·2	60·2	47·9	28·1	79·8	75·7	87·9	88·9	—	—	—	—	—	—	—	—
62·5 ?	—	66·5 ?	62·1	56·7	57·5	45·2	22·7	—	74·1	—	—	—	—	29·5	—	—	—	81°	79°
61·9	85·7	—	—	—	—	48·3	20·0	75·2	—	84·8	88·9	—	—	—	69°·0	67°·8	43°·2	80°·5	78°·5
—	—	—	57·3	53·7	53·2	—	—	78·7	79·2	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
60·6	80·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
60·2	91·4	—	69·1	62·2	62·8	38·2	19·0	74·7	76·2	80·3	84·8	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	27·1	—	77·2	—	—	—	—	—	—	—	—	—	—
61·5	88·9	65·6	55·7	54·3	54·0	38·3	23·2	—	84·1	—	—	—	—	—	69°·8	73°·9	36°·3	—	—
58·9	74·5	—	61·1	58·9	59·7	37·0	—	79·3	79·4	87·9	91·4 ?	—	—	—	—	—	—	82° ?	79°·5
—	75·0	76·3	57·5	51·3	51·7	44·8	33·3	92·1	87·6	98·7	—	—	—	—	64°·0	76°·1	39°·9	87°	—
59·0	90·1	—	61·9	59·0	58·6	—	30·3	—	—	—	—	—	—	—	—	—	—	—	—
55·5 ?	74·6	68·7 ?	58·1	54·5	54·4	43·7	27·1	79·1	79·0	86·7	—	70·9	76·8	36·7	71°·9 ?	66°·9 ?	41°·2 ?	78°·5	78°·0
60·0	81·6 ?	—	54·9	53·0	53·6	45·8	33·7	81·0	—	90·8	—	71·2	81·9	38·5	—	—	—	83°·5	—
57·4	77·5	—	—	—	—	—	35·5	—	—	—	—	—	—	—	—	—	—	—	—
58·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
59·6	82·2	—	—	—	—	—	33·0	—	82·7	—	—	—	—	—	—	—	—	—	—
58·6 ?	—	—	50·2	48·3	49·1	56·3	19·5	77·8	77·1	87·5	—	—	—	—	—	—	—	—	—
62·7	69·6	—	55·8	51·7	52·8	—	19·6	83·8	84·2	86·7	—	—	—	—	—	—	—	—	—
62·5	86·0	74·4	45·9	46·2	45·9	46·5	15·9	85·7	82·9	92·0	—	76·2	82·5	27·1	66°·9	70°·9	42°·2	79°	77°·5
—	—	—	—	—	—	—	33·3	—	83·9	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
58·1 ?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
55·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	31·2	—	—	—	—	—	—	—	—	—	—	—	—
51·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

	Alveolar P∠	*Prosthion P∠*	θ_1	θ_2	Remarks
	—	—	—	—	Aged; sagittal and coronal sutures obliterated, lambdoid closed. Teeth markedly worn and 8 lost before death. JR.
	—	—	—	—	Aged; sagittal and coronal obliterated, lambdoid closed. 9 teeth lost before death. JR.
	84°	82°	—	—	Aged; sagittal and coronal obliterated, lambdoid closed. Teeth moderately worn. Face and base partly defective.
	78°	75°·5	26°·6	12°·5	Coronal closing. Teeth worn. J equal.
	82°·5	81°	28°·7	7°·5	Coronal closing. 1 molar lost before death. Teeth worn. JR.
	—	—	—	—	Aged; all sutures obliterated. 3 teeth lost before death. J equal.
	84°	82°	—	—	Coronal beginning to close. Teeth worn. JR.
	79°?	—	24°·5?	11°·5?	Coronal and sagittal closing. Teeth worn. JR.
	—	—	—	—	Ageing; coronal and sagittal nearly obliterated, lambdoid closed. 2 teeth lost before death. Face largely defective.
	87°	84°?	29°·2	10°·3	Aged; all sutures obliterated. Teeth worn; 2 lost before death. JR.
	86°	84°	22°·6	16°·1	Teeth considerably worn. JL.
	85°	82°·5	26°·1	15°·6	Sagittal and lambdoid partly obliterated, coronal beginning to close. Teeth markedly worn. JR.
	—	—	—	—	Coronal and sagittal beginning to close. JL. Right side of face missing.
	—	—	—	—	Coronal closing. Teeth worn; 1 lost before death. JR.
	81°·5?	80°	29°·5?	12°·5?	Coronal closed, sagittal closing. Teeth worn. J equal.
	79°	76°·5	31°·2	10°·6	Coronal closed, sagittal closing. Teeth worn. JR.
	90°·5	89°·5	26°·0	15°·0	Aged; all sutures obliterated. Teeth moderately worn; 2 lost before death. JR.
	86°	83°	—	—	Ageing; all sutures partly obliterated. Teeth markedly worn. Base defective.
	—	—	—	—	Sagittal closing. Edentulous. JL.
	—	—	—	—	J equal. Face missing.
	76°·5	—	25°·5	12°·2	Coronal and sagittal closed. Teeth worn. J equal.
	—	—	—	—	Ageing; coronal obliterated, sagittal and lambdoid closed. Face, base, and right temporal missing.
	—	—	—	—	Teeth worn; several lost before death. Greater part of brain-box missing.
	79°	77°	26°·5	8°·2	Teeth moderately worn. JL. Face partly defective.
	—	—	—	—	Teeth worn and several lost before death. Vault and base defective.
?	87°·5	86°·5	29°·7?	13°·0?	Coronal closed, sagittal and lambdoid beginning to close. Teeth slightly worn. J equal.
	—	—	—	—	Teeth worn. Nearly all of brain-box missing.
	86°·5	83°	27°·5	15°·9	Coronal and sagittal beginning to close. Teeth worn. JR.
	—	—	—	—	Coronal beginning to close. 11 teeth lost before death. JR.
	—	—	—	—	Teeth slightly worn; 2 lost before death. JR.
	89°·5	87°·5	24°·2	16°·0	Sagittal closed. Teeth markedly worn; 6 molars lost before death. JR.
	83°	79°·5	31°·0	5°·5	Sagittal closing. 3 teeth lost before death. JR.
	80°	79°	32°·4	12°·0	Sagittal closed, coronal closing, lambdoid beginning to close. Teeth hardly worn. JR.
	—	84°·5	—	—	Teeth markedly worn; 6 lost before death. JL.
	85°	82°·5	25°·0	17°·3	Ageing; sagittal obliterated, coronal and lambdoid partly obliterated. Teeth markedly worn. JR.
	—	—	—	—	Aged; sagittal and lambdoid obliterated, coronal nearly obliterated. Teeth worn. JL.
	82°·5	—	29°·3	9°·4	Ageing; sagittal obliterated, coronal and lambdoid nearly obliterated. Teeth worn and 1 lost before death. JR.
	80°·5	78°	32°·0	10°·7	Coronal suture closed. Teeth hardly worn. JR.
	78°	76°·5	25°·8	13°·3	Teeth hardly worn.
	88°	84°	28°·3	15°·1	Ageing; all sutures nearly obliterated. Teeth hardly worn; 1 lost before death. JR.
	—	—	—	—	Aged; all sutures obliterated. Teeth moderately worn; 2 lost before death. J equal.
	80°	78°	28°·3	10°·2	Ageing; coronal and sagittal obliterated, lambdoid nearly obliterated. 3 teeth lost before death. JR.
	—	—	—	—	Sagittal and lambdoid closed, coronal closing. Teeth markedly worn; most lost before death. JR.
	86°	83°	25°·4	13°·6	JR.
	—	—	—	—	Teeth hardly worn. Base defective.
	—	—	—	—	Ageing; all sutures closed. Teeth markedly worn; 9 lost before death. JR.
	81°	79°	—	—	Coronal suture closed. Teeth worn. JL.
	80°·5	78°·5	30°·5	12°·7	Teeth worn. J equal.
	—	—	—	—	Coronal and sagittal partly obliterated. Base defective.
	—	—	—	—	Face missing and base defective.
	—	—	—	—	JR. Part of frontal and face missing.
	—	—	—	—	Teeth markedly worn; 11 lost before death. JL.
	—	—	—	—	Frontal and part of left parietal and face only.
	—	—	—	—	Teeth worn; 4 lost before death. JL.
	82°?	79°·5?	—	—	Sagittal beginning to close. Teeth moderately worn; 6 lost before death. JR.
	87°	—	29°·0	10°·9	Teeth markedly worn; 5 lost before death. JR.
	—	—	—	—	1 or more teeth lost before death. JL.
?	78°·5	78°·0	29°·6?	11°·6?	Ageing; sagittal partly obliterated, coronal and lambdoid closed. Teeth hardly worn and central incisors only lost before death. JR.
	83°·5	—	—	—	Ageing; all sutures closed. Teeth hardly worn and apparently none lost before death. Face partly defective. JR.
	—	—	—	—	Sagittal closing, coronal beginning to close. Calvaria only. JR.
	—	—	—	—	Sagittal closed, coronal beginning to close. Calvaria only. JR.
	—	—	—	—	Sagittal nearly obliterated, coronal nearly closed, lambdoid closing. Most teeth present at death. Face partly defective. J equal.
	—	—	—	—	Aged; sagittal and coronal obliterated, lambdoid nearly obliterated. Most teeth lost before death. Right side of calvaria partly defective. JR.
	—	—	—	—	Ageing; coronal obliterated, sagittal partly obliterated. Teeth worn; 8 lost before death. JL.
	79°	77°·5	34°·1	8°·1	Young adult; third molars not fully erupted. JR.
	—	—	—	—	Face, frontal bone and part of left side of vault only. Most teeth present at death.
	—	—	—	—	Aged; all sutures obliterated. Incomplete vault.
	—	—	—	—	Ageing; sagittal and coronal obliterated. Teeth moderately worn. Incomplete left side of calvaria, occipital bone and part face.
	—	—	—	—	Sagittal obliterated, lambdoid partly obliterated. Incomplete calvaria.
	—	—	—	—	Incomplete frontal and nasal bridge.
	—	—	—	—	Coronal and sagittal partly obliterated. Vault and supra-occipital only.

	Serial No.	Cave No.	Sex	*C*	*P′*	*B″*	*B*	*Biasterionic B*	*F*	*L*	*H′*	*H*	*OH*	*LB*	S_1'	S_2'	*S*
TEITA	58	A 6	♀?	1380	99·0	116·5	128	102·4	185·5	186	133	132	115	100·3	112·9	120·9	9
	59	A 9	,,	1191	92·9	114·5	129	97·8	177	176·5	131	129·5	110	97·0	113·0	110·4	9
	60	E 13	,,	1347	92·0	112·5	129	106·7	181	182·5	126	125·5	106	100·2	109·1	115·7	8
	61	E 16	,,	—	82·5	102·5 ?	128	96·8	178·5	179	—	—	107	—	99·4	112·0	9
	62	E 21	,,	—	90·3	109·5	126	106·1	171·5	171	126·5	126·5	106	89·2	108·5	109·9	8
	63	E 24	,,	—	94·8	114·5	126	98·5	171	172	122	121·5	105	101·4	104·6	103·0	8
	64	E 27	,,	1204	99·8	111·5	134	101·8	177	177	128·5	128	106·5	97·5	106·5	108·4	8
	65	E 45	,,	—	96·3	111	129	105·8	179·5	178	—	—	103·5	—	99·3	109·0	-
	66	E 46	,,	—	96·5	106·5	125·5	104·4	180	178	129	128	113	97·5	106·1	115·4	9
	67	E 60	,,	—	86·8	—	—	—	—	—	—	—	—	—	—	—	-
	68	E 61	,,	—	89·4	111·5	131	97·5 ?	177·5	177	128·5	127	113·5	98·1	106·2	119·3	-
	69	E 64	,,	—	—	—	129·5	99·1 ?	—	—	135	—	—	—	—	114·3	8
	70	F 5	,,	1392	96·6	113·5	130	104·0	186	185·5	130	130	111·5	100·6	108·7	114·0	9
	71	G 4	,,	—	84·5	110·5	139	107·4	185	183·5	130	—	—	94·0	116·4	116·9	9
	72	A 1	♀	1229	90·3	105·5	129·5	105·5 ?	177	176	128	127	105	95·2	105·5	119·3	8
	73	A 3	,,	—	91·9	106	121·5	99·9	171·5	172	119·5 ?	118	104·5	91·0 ?	106·6	108·8	8
	74	A 4	,,	1095	84·6	103	123	97·1	171	169·5	123	121	104·5	91·4	102·4	100·9	8
	75	A 7	,,	1199	92·5	109	129·5	101·8	179	177	125·5	124·5	107	93·3	106·5	112·9	9
	76	A 10	,,	—	93·3	102 ?	125	98·5	174·5	173	—	—	—	—	107·1	—	-
	77	A 11	,,	1353	92·0	113	133·5	106·8	184	184·5	130·5	129·5	112·5	98·5	113·6	119·5	9
	78	B 1	,,	1175	91·2	107·5	125	99·5	175·5	172	125	123·5	108	95·5	105·9	103·4	9
	79	B 2	,,	1109	99·3	114·5	129·5	101·5	169	167·5	130	129	108	96·1	102·2	107·0	-
	80	C 1	,,	—	86·4	100·5	123	101·1	166	167	122·5	122	102·5	94·9	98·5	103·7	8
	81	C 2	,,	1081	86·5	105	116·5	99·4	173·5	173	125	124	108	98·3	111·0	102·0	8
	82	D 1	,,	—	91·0	112 ?	—	—	184	184·5	—	—	—	—	104·6 ?	115·4 ?	9
	83	D 3	,,	1194	93·1	108	122	97·3	184	183	124	122·5	106	96·5	108·0	113·8	9
	84	E 2	,,	1324	98·2	117	133·5	107·4	182	180	124	124	106·5	93·0	110·1	113·0	9
	85	E 4	,,	1194	91·6	108	125·5	102·7	177·5	175·5	124·5	123	108	97·5	107·2	112·4	8
	86	E 8	,,	1258	97·1	116·5	129·5	98·7	180·5	178	111·5	111	100	94·0	99·5	111·0	8
	87	E 9	,,	1100	97·8	100·5	122·5	100·2	179·5	178·5	125	123	102·5	97·1	105·9	109·6	8
	88	E 10	,,	—	91·8	103	124	102·9	177·5	176	130·5	129·5	110·5	—	107·8	110·1	9
	89	E 11	,,	1057	86·6	102·5	125·5	100·4	173	173	125	124	104	95·9	104·9	105·3	8
	90	E 14	,,	1247	94·0	112	123	101·7	182	179	128	126·5	106·5	100·0	107·7	115·1	8
	91	E 22	,,	—	91·3	108	124	102·6	178·5	177	—	123·5	104	—	103·2	113·6	-
	92	E 23	,,	—	84·4	100·5	121·5	95·5	163·5	163·5	119	118·5	101	90·0	99·8	101·2	8
	93	E 25	,,	1161	95·4	109·5	129	95·8	170	168	125·5	126	110	91·0	106·3	109·7	8
	94	E 26	,,	—	92·6	105	125·5	101·1	168·5	166·5	123·5	123	107·5	93·3	101·7	107·1	9
	95	E 29	,,	—	92·0	106·5	133·5	107·7	172	172·5	—	—	—	—	102·9	106·8	8
	96	E 31	,,	—	—	110	133	103·5	180·5 ?	—	—	—	—	—	—	112·4	10
	97	E 32	,,	—	97·3	113·5	122·5	—	176	175·5	—	—	—	—	108·5	108·8	-
	98	E 33	,,	—	93·5	105·5	121	—	—	—	—	—	—	—	105·4	102·8	-
	99	E 35	,,	1129	91·8	106·5	123	97·5	170·5	173	124·5	123·5	105	96·9	108·0	108·3	9
	100	E 37	,,	1158	90·2	106	124·5	99·5	180	180	127·5	126	112	100·1	113·6	113·2	9
	101	E 38	,,	965	91·0	105	120·5	95·9	177·5	177	120·5	119	100	—	—	109·2	8
	102	E 49	,,	—	93·0	110	128	102·1	185·5	184	127	—	—	102·4	106·3	110·6	9
	103	E 50	,,	—	92·2	108	125·5	99·2 ?	170	171	—	—	—	—	101·8	101·5	-
	104	E 54	,,	—	94·0	110	131	101·3	181	180	128	127	107·5	104·4	104·6	106·4	8
	105	E 56	,,	—	98·4	106·5	127	100·1	178·5	178	128	127	107	100·3	104·4	112·5	9
	106	E 57	,,	—	93·7	107 ?	126·5	101·4 ?	—	—	—	—	110	—	104·8	—	-
	107	E 59	,,	—	92·0	101	123·5	102·9	169·5	169·5	—	—	109·5	—	107·1	101·4	9
	108	E 65	,,	—	86·9	—	—	—	—	—	—	—	—	—	—	—	-
	109	E 67	,,	—	90·6	—	—	—	—	—	—	—	—	—	—	—	-
	110	F 1	,,	1097	89·8	106	120	100·9	171	171·5	122·5	122	106	99·9	102·7	108·1	9
	111	F 2	,,	—	90·7	103·5	119	96·3	169	169·5	124	123	103·5	97·0	102·7	98·2	8
	112	G 9 *a*	,,	—	96·6	—	—	—	177	175	—	—	—	—	107·4	103·6	-
	113	H 1	,,	1203	93·3	112·5	129·5	102·0	174·5	172	131·5	131·5	112	94·4	109·6	109·8	9
	114	J 1	,,	1181	98·5	101·5	126·5	106·7	174·5	175	126	125	106	99·8	104·1	111·7	9
	115	J 4	,,	1214	96·5	103·5	120	91·6 ?	178	177	130·5	129·5	109·5	95·3	112·7	107·0	-
	116	J 5	,,	1180	95·0	106·5	123	99·8	176·5	177·5	124·5	123·5	103·5	99·5	108·1	104·9	9
	117	K 2	,,	—	94·1	116·5	131	104·8	177	176	126	124·5	113	95·5	115·5	101·2	9
	118	K 4	,,	1277	94·5	107	125·5	99·2	181	180·5	131	130	112	101·5	112·1	108·6	9
	119	L 1	,,	1188	93·7	108·5	124·5	98·1	174·5	174·5	120·5	121	103·5	94·0	95·9	110·8	9
	120	L 3	,,	1234	94·6	109	129	101·9	179	178	125	124	106·5	94·8	107·7	109·5	9
	121	L 5	,,	1107	89·5	106·5	125	96·8	172	170	123	122	106	94·4	104·3	102·3	8
	122	L 6	,,	1124	85·3	103·5	123·5	100·5	174	174	125·5	126	111·5	94·8	104·6	109·9	8
	123	L 7	,,	—	—	124	138	—	—	—	—	—	—	—	—	117·9	-
NAISHI RIVER	125	—	,,	1276	98·5	112	129	103·5	177·5	178·5	124·5	123	104·5	88·8	110·1	114·0	9

S_1'	S_2'	S_3'	S_1	S_2	S_3	S	U	*Glabella U*	Q'	*Bregmatic Q'*	$G'H$	GL	GB	J	PH	NH, R	NH, L	NH'	NB
2·9	120·9	94·0	129	132·5	110	371	509	511	304·5	305	—	—	101·7	131	—	51·0	50·0	44·9	29·3
3·0	110·4	94·7	131	122·5	113	367	495	495	302·5	302	62·1	97·8	91·8 ?	123 ?	—	45·8	47·8	45·2 ?	24·9
9·1	115·7	88·4	127	129	105	362	505·5	509·5	291·5	290	67·0	98·0	102·4	130·5	—	51·0	51·3	—	27·8
9·4	112·0	95·0	111·5	121·5	117	350	491	497	296	289	—	—	—	124·5	—	52·0	51·0	47·7	26·2
8·5	109·9	89·4	128	124·5	106	359	479	479	285	285	64·0	83·2	88·5	125	16·0	47·2	47·9	47·5	27·0
4·6	103·0	81·3	119	115·5	97	331·5	476	477	285	282	67·0	97·6	93·4	123·5	19·8 ?	50·1	50·7	47·1	28·5
6·5	108·4	89·7	122	121·5	106	350	503	505·5	291	288	—	—	91·7	124 ?	—	44·9	44·9	41·8	30·1
9·3	109·0	—	115·5	121	—	—	495	496	282	283	60·4	—	—	—	18·3	45·0	—	42·5	—
6·1	115·4	94·3	122	127	111·5	361	498	499	301·5	296·5	63·1	92·3	99·7	—	—	49·7	50·1	45·0	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	27·2
6·2	119·3	—	123	135·5	—	—	496	493	312	305·5	67·3	94·6	—	—	22·2	59·5	49·0	45·8	26·3
—	114·3	85·7 ?	—	127 ?	100 ?	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8·7	114·0	95·1	124	129·5	116	369	510	507	299	296	71·8	96·2	98·2	126	—	50·2	50·2	47·8	—
6·4	116·9	92·4	134·5	130·5	114	379	505	504	—	302·5	—	—	—	114·5 ?	—	—	—	—	—
5·5	119·3	85·4	120	137	101	358	490	488	281·5	279·5	68·8	95·5	95·2	121 ?	—	43·0	43·2	41·0	—
6·6	108·8	82·7	123·5	123	96·5	343	482	483·5	280	278	71·3 ?	96·5 ?	89·1 ?	118 ?	25·5 ?	47·0	47·9	45·4	29·0
2·4	100·9	87·4	118	111	108	336	469·5	468	283·5	281	57·1	93·7	90·2	122	19·0	40·9	41·0	38·9	24·7
6·5	112·9	92·1	123·5	125	110	358	492	489	295·5	294	—	—	—	—	—	46·0	45·6	43·0	25·0
7·1	—	—	125 ?	—	—	—	481	477	—	298·5	—	—	—	—	—	—	—	—	—
3·6	119·5	93·0	129	136	111	376	508	509	303	301	70·2	94·9	90·4	123·5	23·8	51·4	50·9	46·2	28·9
5·9	103·4	93·6	122	113	116	350·5	484	483	288·5	286	—	—	90·0	125·5	—	43·9	43·9	41·3	26·2
2·2	107·0	—	116·5	120·5	—	—	481	483	294·5	292	—	—	—	—	—	47·5	—	—	—
8·5	103·7	83·7	113	116	98	327	468	471	278·5	273	56·9	89·9	—	—	19·5	42·0	42·8	37·9	26·0
1·0	102·0	82·7	127	113·5	101	343	476	478	283·5	281·5	—	—	92·0	117	—	53·0	52·0	47·4	27·2
4·6 ?	115·4 ?	94·0 ?	119 ?	128 ?	115 ?	362	506 ?	—	—	—	69·8	—	96·9	—	18·8	54·1	54·3	51·0	30·8
8·0	113·8	93·2	131	126·5	110	367·5	495	493	286	285	61·1	95·0	—	—	—	47·0	45·5	43·1	27·4
0·1	113·0	91·3	132·5	122	111	366	508·5	509	302	299	62·0	89·2	99·2	—	—	49·0	47·0	40·3	29·4
7·2	112·4	83·6	124	128	101·5	353	490	490	290	288	—	—	99·2	124·5	—	47·9	47·1	45·0	27·0
9·5	111·0	86·0	125·5	126·5	106	358	505	501	282·5	281·5	61·1	89·8 ?	86·2	126	—	43·8	44·8	—	22·0
5·9	109·6	89·4	121·5	119·5	109	350	484·5	486	276	276	62·7	95·1	81·9	118·5	19·0	46·8	46·8	44·4	26·0
7·8	110·1	91·7	124	122	110·5	357	490·5	493	290·5	287	—	96·1 ?	95·8	—	18·9 ?	—	—	—	26·5
4·9	105·3	85·7	120	116·5	104·5	341	478	479·5	280	278	73·7	97·7	91·2	125	—	48·8	48·6	46·5	26·3
7·7	115·1	89·3	128	129	104	361	498	495	285·5	285	67·6	102·3	96·7	124	20·3	48·2	49·1	46·8	24·5
3·2	113·6	—	120	125	—	—	489	487	281·5	280	—	—	89·0	—	—	—	—	—	—
9·8	101·2	85·0	113	114·5	100	328·5	457	458	274	271	65·8	93·0	90·1	121·5	20·0	47·3	46·7	45·9	26·5
6·3	109·7	82·8	125	128	100	353	477	477	298	289·5	59·0	90·6	91·3	—	19·0	40·9	41·4	39·5	26·1
1·7	107·1	91·4	116·5	122	107·5	347	470	473	288·5	283	—	—	89·5	—	—	45·2	44·5	41·8	25·9
2·9	106·8	86·5	120	117	105	343	486	489	—	285	—	—	—	—	—	—	—	—	—
—	112·4	100·6	—	121·5	119·5	—	496	—	—	304 ?	—	—	—	—	—	—	—	—	—
8·5	108·8	—	128	122	—	—	486 ?	486 ?	—	—	—	—	—	—	—	—	—	—	—
5·4	102·8	—	125	113·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8·0	108·3	94·4	123·5	126	106	354·5	476	478	280·5	278	60·5	95·8	93·2	119	20·8 ?	43·3	43·1	41·1	26·5
3·6	113·2	90·6	132	126	107·5	365	494·5	494	298	296	—	97·2	97·1	122·5	—	47·9	47·5	45·0	26·0
—	109·2	82·1	—	119	101	—	482	483·5	271	270·5	—	101·0	—	117·5 ?	—	—	—	—	—
6·3	110·6	94·2	125	125	114·5	365	504	500	—	295	—	—	—	—	—	—	—	—	—
1·8	101·5	—	115	111·5	—	—	480	482	—	284	—	—	—	121·5	—	—	—	—	—
4·6	106·4	87·6	120	120	111·5	351·5	496	497	296·5	294·5	59·8	99·8	96·2	124·5	16·2	47·2	47·7	44·5	30·3
4·4	112·5	93·9	118	125	113·5	356·5	495·5	491·5	286	284	65·1	93·9	106·4	128	18·4	49·9	51·0	46·9	29·0
4·8	—	—	121	—	—	—	—	—	293·5	291	62·1	—	94·1	120·5	—	49·7	47·8	45·0	26·6
7·1	101·4	95·3	120·5	110	112·5	342·5	475·5	479·5	282	280	—	—	99·2	—	—	49·6	49·0	—	27·0
—	—	—	—	—	—	—	—	—	—	—	—	—	81·6	—	—	44·1	44·0	42·5	22·3
—	—	—	—	—	—	—	—	—	—	—	—	—	89·9	—	—	—	—	—	28·5
2·7	108·1	90·5	115·5	122	106	343·5	474	475	277	275	—	—	101·1	121·5	—	42·9	42·6	41·2	30·9
2·7	98·2	86·0	119·5	109	105·5	333	468	466 ?	278·5	276	—	—	94·1	—	—	44·3	45·1	42·5	26·2
7·4	103·6	—	129·5	113	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26·8
0·6	109·8	95·0	125	121·5	111·5	357	482	480	296	293·5	61·0 ?	90·8 ?	97·0	120·5	18·2 ?	46·0	46·0	44·0	28·2
0·1	111·7	90·8	117	125·5	105	347	482	482	287	283	60·2	97·0	—	—	16·3	46·6	46·7	44·0	29·0
2·7	107·0	—	135	117	—	—	483	481	284	283·5	—	—	—	117·5	—	45·5	45·2	44·7	25·8
0·8·1	104·9	93·0	125	116	109	350	487	—	283	280	—	—	—	—	—	49·8	49·5	45·0	28·8
5·5	101·2	99·9	135	109	118	362	497	498	307·5	306	—	—	96·4	123·5	—	46·8	47·2	44·0	25·8
2·1	108·6	94·0	130	121	113·5	364	494	491	301	296	58·8	96·7	95·7	129	—	44·1	45·1	41·9	27·9
5·9	110·8	91·8	110	123	112·5	345·5	487	—	285·5	282	62·5	90·4	—	—	18·0	46·2	46·6	44·3	24·5
0·7·7	109·5	95·9	126·5	122·5	114	363·5	497	—	290	288	—	—	—	—	—	45·2	46·0	—	—
0·4·3	102·3	89·7	123	117	105·5	345·5	479·5	478	292	285	61·3	94·7	97·1	125	21·7	44·1	42·9	39·2	25·4
0·4·6	109·9	89·1	118	123	106·5	347	478·5	—	297	289	63·5	101·8	96·8	124	21·7	48·0	47·0	42·0	29·0
—	117·9	—	—	132·5	—	—	510 ?	—	—	—	—	—	—	—	—	—	—	—	—
10·1	114·0	97·9	126·5	127	119	371·5	497	498	284	284	—	—	92·1	123·5	—	48·5	48·3	47·2	23·9

TABLE XIV *Individual Measurements of Female Sk*

PH	*NH, R*	*NH, L*	*NH'*	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	*O_1R*	*O_1L*	*$O_1'R$*	*$O_1'L$*	*Lacr. O_1R*	*O_2R*	*O_2L*	*G_1'*	*G_1*	*G_2*
—	51·0	50·0	44·9	29·5	10·8	25·9	35·5	2·8	11·0	44·9	44·0	39·2	40·6	—	34·6	34·3	48·5 ?	54·5 ?	—
—	45·8	47·8	45·2 ?	24·9	10·2	25·0	35·5	1·1	8·0	42·0	40·1	35·0 ?	34·5	—	32·7	33·1	46·6	50·0	—
—	51·0	51·3	—	27·8	9·8	25·1	36	1·6	9·9	42·4	41·1	38·8	38·4	37·5	31·7	33·0	46·9	50·1	—
—	52·0	51·0	47·7	26·2	12·4	24·1	37	4·7	15·1	43·9	43·1	38·3	37·7	37·0	34·2	33·6	—	—	—
16·0	47·2	47·9	47·5	27·0	9·1	24·8	31·5	3·0	11·3	41·0	40·2	36·5	37·0	36·5	33·8	34·0	42·3	43·9	38·0
19·8 ?	50·1	50·7	47·1	28·5	—	—	—	1·5	9·9	42·0	41·0	—	—	—	32·0	32·5	44·9	46·9	41·5
—	44·9	44·9	41·8	30·1	—	—	—	2·0	8·9	41·8	41·8	—	—	—	33·0	31·9	—	—	40·9 ?
18·3	45·0	—	42·5	—	—	—	—	—	—	41·7	—	—	—	—	31·0	—	—	—	—
—	49·7	50·1	45·0	—	—	—	—	2·6	14·6	43·2	43·0	—	—	—	31·0	32·0	42·8	46·3	37·5
—	—	—	—	27·2	—	—	—	2·8	10·7	—	—	40·9	—	—	—	33·5	—	—	33·2
22·2	59·5	49·0	45·8	26·3	—	—	—	2·2	11·9	41·8	41·0	38·3	—	—	31·2	30·6	45·3	51·2	35·2
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	50·2	50·2	47·8	—	—	—	—	2·6	11·0	43·2	42·5	—	—	—	32·6	—	45·0	—	40·3
—	—	—	—	—	—	—	—	2·6	10·8	—	—	—	—	—	—	—	—	—	—
—	43·0	43·2	41·0	—	—	—	—	—	10·8	41·2	—	—	—	—	34·0	33·7	45·9	51·2	36·2
25·5 ?	47·0	47·9	45·4	29·0	—	—	—	1·1	10·7	41·9	42·0	—	—	—	33·7	34·3	47·0	—	38·0
19·0	40·9	41·0	38·9	24·7	6·9	23·0	30	0·7	10·0	41·7	40·8	37·9	37·1	37·3	32·0	33·6	42·3	46·0	—
—	46·0	45·6	43·0	25·0	—	—	—	1·2	8·7	42·0	42·0	—	39·1	—	32·5	32·0	—	—	—
—	—	—	—	—	—	—	—	—	—	42·0	—	—	—	—	—	—	—	—	—
23·8	51·4	50·9	46·2	28·9	9·9	22·1	31·5	2·1	8·6	43·0	41·9	37·3	37·0	—	35·0	36·0	46·0	47·0	—
—	43·9	43·9	41·3	26·2	8·3	22·0	31	1·4	9·7	42·5	43·6	39·3	40·1	—	34·9	33·8	—	—	—
—	47·5	—	—	—	—	—	—	—	—	42·8	—	—	—	—	30·2	—	—	—	—
19·5	42·0	42·8	37·9	26·0	—	—	—	2·1	10·3	40·7	40·1	—	36·2	—	31·0	32·0	41·9	46·8	—
—	53·0	52·0	47·4	27·2	—	—	—	2·7	13·0	43·0	42·0	—	—	—	32·7	33·3	—	—	—
18·8	54·1	54·3	51·0	30·8	—	—	—	1·7	9·9	42·9	42·2	37·8 ?	—	37·1 ?	32·8	35·0	—	—	—
—	47·0	45·5	43·1	27·4	—	—	—	2·6	13·0	41·3	—	—	—	—	35·9	—	42·9	48·2	—
—	49·0	47·0	40·3	29·4	—	—	—	0·8	16·0	41·2	42·0	—	—	—	—	33·0	—	—	—
—	47·9	47·1	45·0	27·0	10·8	25·8	35·5	1·5	11·2	41·0	40·8	37·0	37·0	37·0	34·5	34·5	45·9 ?	49·0 ?	—
—	43·8	44·8	—	22·0	9·9	20·7	32	1·4	7·2	41·7	40·9	36·5	37·5	—	36·0	35·5	44·0	45·8	38·2
19·0	46·8	46·8	44·4	26·0	8·1	20·5	31	1·0	7·7	40·9	41·0	37·7 ?	—	—	34·8	34·5	42·4	45·0	34·9
18·9 ?	—	—	—	26·5	—	—	—	—	—	42·3	42·0	—	—	—	33·8	33·4	—	—	—
—	48·8	48·6	46·5	26·3	7·9	21·1	29	1·7	9·8	43·2	42·0	40·8	39·8	38·3	35·2	35·5	51·0	55·8	—
20·3	48·2	49·1	46·8	24·5	9·1	24·2	33	2·5	10·5	41·1	42·0	38·9	38·2	37·2 ?	33·5	—	—	—	41·0
—	—	—	—	—	—	—	—	3·1	9·7	41·0	42·8	37·3 ?	—	—	34·0	33·8	—	—	34·0
20·0	47·3	46·7	45·9	26·5	—	—	—	1·2	5·9	40·0	39·2	35·8	—	35·0	35·2	34·2	44·0	48·2	—
19·0	40·9	41·4	39·5	26·1	9·8	21·0	33	1·0	5·6	40·9	41·9	37·3	36·8	—	30·8	30·2	40·0	42·9	32·0
—	45·2	44·5	41·8	25·9	8·0	22·1	31	0·7	6·0	41·0	40·1	38·0	36·0	36·7	30·8	30·1	40·0	45·1	35·0
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
20·8 ?	43·3	43·1	41·1	26·5	10·1	23·0	34	3·2	11·8	41·9	41·3	36·2	37·9	36·2	33·3	32·9	45·9	50·0	35·8
—	47·9	47·5	45·0	26·0	8·9	22·0	30	2·0	10·0	42·8	42·7	39·8	39·2	39·0	34·0	34·0	44·5	48·9	37·8
—	—	—	—	—	—	—	—	—	—	39·0 ?	38·2 ?	—	—	—	—	34·0	—	—	—
—	—	—	—	—	—	—	—	2·0	9·0	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
16·2	47·2	47·7	44·5	30·3	—	—	—	—	14·2 ?	45·0	46·0	42·0	—	—	37·4	37·9	46·8	52·5	41·1
18·4	49·9	51·0	46·9	29·0	—	—	—	2·6	13·5	41·2	42·0	37·5	37·8	—	35·3	34·2	—	—	—
—	49·7	47·8	45·0	26·6	—	—	—	—	—	42·2	42·0	37·7	—	—	32·5	32·0	—	—	—
—	49·6	49·0	—	27·0	—	—	—	3·4	12·0	44·2	43·1	—	—	—	34·0	35·1	—	—	40·0
—	44·1	44·0	42·5	22·3	—	—	—	3·5	11·0	40·5	40·0	36·1	—	—	32·1	31·1	—	—	34·9
—	—	—	—	28·5	—	—	—	0·9	9·6	45·4	43·8	—	—	—	34·8	34·5	—	—	35·8 ?
—	42·9	42·6	41·2	30·9	—	—	—	0·7	5·3	42·0	41·9	—	—	—	34·0	34·0	49·1 ?	53·3 ?	—
—	44·3	45·1	42·5	26·2	7·7	24·6	31	1·7	8·3	—	40·0	—	38·1	—	—	30·5	43·5	46·2	38·0
—	—	—	—	26·8	—	—	—	0·7	6·0	42·9	44·2	39·8	—	—	36·0	32·0	—	—	26·0 ?
18·2 ?	46·0	46·0	44·0	28·2	10·2	26·0	37	1·0	11·7	40·0	—	36·8	—	35·9	30·9	30·8	46·0	49·0	36·5
16·3	46·6	46·7	44·0	29·0	10·0	25·3	36	2·2	11·6	—	43·1	—	39·0	—	33·5	33·3	45·8	50·5	39·7
—	45·5	45·2	44·7	25·8	—	—	—	0·5	9·6	37·5	39·1 ?	—	—	—	32·5	—	45·6	48·7	—
—	49·8	49·5	45·0	28·8	—	—	—	1·2	14·0	—	42·0	—	—	—	—	31·0	37·8	40·8 ?	—
—	46·8	47·2	44·0	25·8	—	—	—	1·2	11·0	43·2	41·2	39·8	39·0	—	36·6	36·9	—	—	—
—	44·1	45·1	41·9	27·9	11·5	25·5	39	1·4	10·5	42·0	41·5	39·0	38·8	37·2	29·2	30·2	45·9	50·2	—
18·0	46·2	46·6	44·3	24·5	9·1	27·1	37·5	3·3	12·0	—	40·0	—	35·4	—	—	32·8	41·9	45·7	—
—	45·2	46·0	—	—	—	21·8	—	—	—	40·5 ?	—	37·3	—	—	30·7	31·1 ?	—	—	—
21·7	44·1	42·9	39·2	25·0	8·6	24·9	33	—	—	42·0	42·0	39·7	39·0	—	33·7	34·8	44·2	49·1	38·0
21·7	48·0	47·0	42·0	29·0	—	—	—	2·2	7·1	—	41·5	—	37·5 ?	—	—	32·5	48·2	51·3	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	48·5	48·3	47·2	23·9	9·7 ?	26·0 ?	34·5 ?	2·2	8·4	44·2	43·0	39·9	39·7	38·1 ?	35·0	33·6	45·5	51·0	—

Female Skulls from Kenya Colony.

G_1	G_2	EH	fml	fmb	100 $\frac{B}{L}$	100 $\frac{H'}{L}$	100 $\frac{H}{L}$	100 $\frac{B}{H'}$	100 $\frac{B}{H}$	100 $\frac{(B-H')}{L}$	$Oc.\ I.$	100 $\frac{fmb}{fml}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH'}$	100 $\frac{NB}{NH'}\ R$	100 $\frac{NB}{NH'}\ L$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	1 $\frac{O}{O}$
54·5 ?	—	—	38·8	33·0	68·8	71·5	71·0	96·2	97·0	−2·7	62·7	85·1	—	65·7	57·8	59·0	41·7	25·5	7
50·0	—	—	31·6	27·0	73·1	74·2	73·4	98·5	99·6	−1·1	60·6	85·4	67·6 ?	55·1 ?	54·4	52·1	40·8	13·8	7
50·1	—	—	39·0	30·4	70·7	69·0	68·8	102·4	102·8	1·6	61·0	77·9	65·4	—	54·5	54·2	39·0	16·2	7
—	—	—	—	28·5	71·5	—	—	—	—	—	57·9	—	—	54·9	50·4	51·4	51·5	31·1	7
43·9	38·0	12·0	32·5	28·7	73·7	74·0	74·0	99·6	99·6	−0·3	61·1	88·3	72·3	56·8	57·2	56·4	36·7	26·5	8
46·9	41·5	10·2	35·3	27·8	73·3	70·9	70·6	103·3	103·7	2·3	60·5	78·8	71·7	60·5	56·9	56·2	—	15·2	7
—	40·9 ?	—	42·8	33·1	75·7	72·6	72·3	104·3	104·7	3·1	61·5	77·3	—	72·0	67·0	67·0	—	22·5	7
—	—	—	—	—	72·5	—	—	—	—	—	—	—	—	—	—	—	—	—	7
46·3	37·5	10·5	38·1	28·0	70·5	72·5	71·9	97·3	98·0	−2·0	61·5	73·5	63·3	—	—	—	—	19·2	7
—	33·2	7·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26·2	—
51·2	35·2	12·5	—	26·0	74·0	72·6	71·8	101·9	103·1	1·4	—	—	—	57·4	53·1	53·7	—	18·5	7
—	—	—	34·4	—	—	—	—	95·9	—	—	63·0 ?	—	—	—	—	—	—	—	—
—	40·3	16·2	39·5	28·3	70·1	70·1	70·1	100·0	100·0	0·0	58·6	71·6	73·1	—	—	—	—	23·6	7
—	—	—	37·2	29·7	75·7	70·8	—	106·9	—	4·9	57·8	79·8	—	—	—	—	—	24·1	—
51·2	36·2	14·1	32·4	27·8	73·6	72·7	72·2	101·5	102·0	0·9	61·4	85·8	72·3	—	—	—	—	—	8
—	38·0	15·1	36·8	28·8	70·6	69·5 ?	68·6	101·7 ?	103·0	1·2 ?	63·0	78·3	80·0 ?	63·9	61·7	60·5	—	10·3	8
46·0	—	—	37·1	26·7	72·6	72·6	71·4	100·0	101·7	0·0	57·7	72·0	63·3	63·5	60·4	60·2	30·0	7·0	7
—	—	—	36·9	28·7	73·2	70·9	70·3	103·2	104·0	2·3	60·5	77·8	—	58·1	54·3	54·8	—	13·8	7
—	—	—	—	—	72·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—
47·0	—	—	37·1	27·8	72·4	70·7	70·2	102·3	103·1	1·6	60·5	74·9	77·7	62·6	56·2	56·8	44·8	24·4	8
—	—	—	36·1	29·9	72·7	72·7	71·8	100·0	101·2	0·0	57·6	82·8	—	63·4	59·7	59·7	37·7	14·4	8
—	—	—	—	—	77·3	77·6	77·0	99·6	100·4	−0·3	—	—	—	—	—	—	—	—	7
46·8	—	—	36·0	28·7	73·7	73·4	73·1	100·4	100·8	0·3	62·6	79·7	—	68·6	61·9	60·7	—	20·4	7
—	—	—	38·0	26·0	67·3	72·3	71·7	93·2	94·0	−4·9	58·6	68·4	—	57·4	51·3	52·3	—	20·8	7
—	—	—	—	30·9	—	—	—	—	—	—	58·5 ?	—	72·0	60·4	56·9	56·7	—	17·2	7
48·2	—	—	33·5	26·6	66·7	67·8	66·9	98·4	99·6	−1·1	61·7	79·4	—	63·6	58·3	60·2	—	20·0	8
—	—	—	36·8	29·0	74·2	68·9	68·9	107·7	107·7	5·3	58·9	78·8	62·5	73·0	60·0	62·6	—	5·0	—
49·0 ?	—	—	34·0	27·0	71·5	70·9	70·1	100·8	102·0	0·6	68·4	79·4	—	60·0	56·4	57·3	41·9	13·4	8
45·8	38·2	10·3	34·6	30·0	72·8	62·6	62·4	116·1	116·7	10·1	57·9	86·7	70·9	—	50·2	49·1	47·8	19·4	8
45·0	34·9	11·2	38·0	27·9	68·6	70·0	68·9	98·0	99·6	−1·4	58·7	73·4	76·6	58·6	55·6	55·6	39·5	13·0	8
—	—	—	35·0	27·0	70·5	74·1	73·6	95·0	95·8	−3·7	59·6	77·1	—	—	—	—	—	—	7
55·8	—	—	37·7	27·2	72·5	72·3	71·7	100·4	101·2	0·3	58·7	72·1	80·8	56·6	53·9	54·1	37·4	17·3	8
—	41·0	13·1	34·0	27·5	68·7	71·5	70·7	96·1	97·2	−2·8	63·2	80·9	69·9	52·4	50·8	49·9	37·6	23·8	8
—	34·0	10·3	—	—	70·1	—	69·8	—	100·4	—	—	—	—	—	—	—	—	32·0	8
48·2	—	—	35·5	26·8	74·3	72·8	72·5	102·1	102·5	1·5	62·1	75·5	73·0	57·7	56·0	56·7	—	20·3	8
42·9	32·0	11·2	34·8	30·0	76·8	74·7	75·0	102·8	102·4	2·1	59·4	79·6	64·6	66·1	63·8	63·0	46·7	17·9	7
45·1	35·0	9·0	32·1	29·4	75·4	74·2	73·9	101·6	102·0	1·2	62·1	91·6	—	62·0	57·3	58·2	36·2	11·7	7
—	—	—	—	28·0	77·4	—	—	—	—	—	59·0	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	61·0	—	—	—	—	—	—	—	—
—	—	—	—	—	69·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
50·0	35·8	10·8	30·7	26·8	71·1	72·0	71·4	98·8	99·6	−0·9	69·3	87·3	64·9	64·5	61·2	61·5	43·9	27·1	7
48·9	37·8	11·3	34·7	29·0	69·2	70·8	70·0	97·6	98·8	−1·7	61·0	83·6	—	57·8	54·3	54·7	40·5	20·0	7
—	—	—	33·0	24·1	68·1	68·7	67·3	100·0	101·3	0·0	58·1	73·0	—	—	—	—	—	—	—
—	—	—	38·8	28·9	69·6	69·0	—	100·8	—	0·5	59·0	74·5	—	—	—	—	—	22·2	—
—	—	—	—	—	73·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—
52·5	41·1	8·8	33·0	29·0	72·8	71·1	70·6	102·3	103·1	1·7	56·1	87·9	61·1	68·1	64·2	63·5	—	—	8
—	—	—	33·0	28·2	71·3	71·9	71·3	99·2	100·0	−0·6	59·4	85·5	61·2	61·8	58·1	56·9	—	19·3	8
—	—	—	—	—	—	—	—	—	—	—	—	—	60·0	59·1	53·5	55·6	—	—	7
—	40·0	—	—	32·5	72·9	—	—	—	—	—	61·7	—	—	—	54·4	55·1	—	28·3	7
—	34·9	—	—	—	—	—	—	—	—	—	—	—	—	52·5	50·6	50·7	—	31·8	7
—	35·8 ?	8·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9·4	7
53·3 ?	—	—	34·5	30·3	70·0	71·4	71·1	98·0	98·4	−1·5	62·6	87·8	—	75·0	72·0	72·5	—	13·2	8
46·2	38·0	10·3	34·8	27·3	70·2	73·2	72·6	96·0	96·7	−2·9	58·2	78·4	—	61·6	59·1	58·1	31·3	20·5	—
—	26·0 ?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11·7	8
49·0	36·5	11·8	34·8	28·3	75·3	76·5	76·5	98·5	98·5	−1·2	62·3	81·3	62·9 ?	64·1	61·3	61·3	39·2	8·5	7
50·5	39·7	7·0	35·7	32·3	72·3	72·0	71·4	100·4	101·2	0·3	64·2	90·5	—	65·9	62·2	62·1	39·5	19·0	—
48·7	—	—	—	—	67·8	73·7	73·2	92·0	92·7	−5·9	—	—	—	57·7	56·7	57·1	—	5·2	8
40·8 ?	—	—	36·4	31·3	69·3	70·1	69·6	98·8	99·6	−0·8	62·4	86·0	—	64·0	57·8	58·2	—	8·6	—
—	—	—	34·5	29·0	74·4	71·6	70·7	104·0	105·2	2·8	61·6	84·1	—	58·6	55·1	54·7	—	10·9	8
50·2	—	—	36·8	30·1	69·5	72·6	72·0	95·8	96·5	−3·0	59·5	81·8	61·4	66·6	63·3	61·9	45·1	13·3	6
45·7	—	—	32·5	26·9	71·3	69·1	69·3	103·3	102·9	2·3	58·4	82·8	—	55·3	53·0	52·6	33·6	27·5	—
—	—	—	34·9	25·8	72·5	70·2	69·7	103·2	104·0	2·2	60·9	73·9	—	—	—	—	—	—	7
49·1	38·0	10·7	36·6	30·8	73·5	72·4	71·8	101·6	102·5	1·2	62·1	84·2	63·1	63·8	56·7	58·3	34·5	—	8
51·3	—	—	38·0	28·8	71·0	72·1	72·4	98·4	98·0	−1·1	60·4	75·8	65·6	69·0	60·4	61·7	—	31·0	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
51·0	—	—	34·5	30·0	72·3	69·8	68·9	103·6	104·9	2·5	58·9	87·0	—	50·6	49·3	49·5	37·3 ?	26·1	7

100 $\frac{mb}{ml}$	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH}$	100 $\frac{NB}{NH'}$, R	100 $\frac{NB}{NH'}$, L	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1'}$, R	100 $\frac{O_2}{O_1'}$, L	100 $\frac{O_2}{O_1''}$, R	100 $\frac{O_2}{Lacr.\ O_1}$, R	100 $\frac{G_2}{G_1}$	100 $\frac{G_2}{G_1'}$	100 $\frac{EH}{G_2}$	$N\angle$	$A\angle$	$B\angle$	*Alveolar* $P\angle$	*Prosthion* $P\angle$
5·1	—	65·7	57·8	59·0	41·7	25·5	77·1	78·0	88·3	—	—	—	—	—	—	—	—	—
5·4	67·6 ?	55·1 ?	54·4	52·1	40·8	13·8	77·9	82·5	93·4 ?	—	—	—	—	72°·0	70°·9	37°·1	82°	80°
7·9	65·4	—	54·5	54·2	39·0	16·2	74·8	80·3	81·7	84·5	—	—	—	68°·4	72°·1	39°·5	85°	83°
—	—	54·9	50·4	51·4	51·5	31·1	77·9	78·0	89·3	92·4	—	—	—	—	—	—	—	—
8·3	72·3	56·8	57·2	56·4	36·7	26·5	82·4	84·6	92·6	92·6	86·6	89·8	31·6	63°·3	73°·4	43°·3	84°·5	82°·5
8·8	71·7	60·5	56·9	56·2	—	15·2	76·2	79·3	—	—	88·5	92·4	24·6	67·1°	73°·7	39°·2	86°	84°·5
7·3	—	72·0	67·0	67·0	—	22·5	78·9	76·3	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	74·3	—	—	—	—	—	—	—	—	—	80°	76°
3·5	63·3	—	—	—	—	19·2	71·8	74·4	—	—	81·0	87·6	28·0	66°·3	75°·1	38°·6	85°·5	85°
—	—	—	—	—	—	26·2	—	—	—	—	—	—	22·6	—	—	—	—	—
—	—	57·4	53·1	53·7	—	18·5	74·6	74·6	81·5	—	68·8	77·7	35·5	66°·3	72°·2	41°·5	89°	86°
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
1·6	73·1	—	—	—	—	23·6	75·5	—	—	—	—	89·6	40·2	65°·5	71°·9	42°·6	85°	84°
9·8	—	—	—	—	—	24·1	—	—	—	—	—	—	—	—	—	—	—	—
5·8	72·3	—	—	—	—	—	82·5	—	—	—	70·7	78·9	39·0	69°·0	68°·9	42°·1	77°·5	75°·5
8·3	80·0 ?	63·9	61·7	60·5	—	10·3	80·4	81·7	—	—	—	80·9	39·7	71°·8 ?	63°·5 ?	44°·7 ?	—	—
2·0	63·3	63·5	60·4	60·2	30·0	7·0	76·7	82·4	84·4	85·8	—	—	—	74°·0	70°·0	36°·0	80°·5	77°·5
7·8	—	58·1	54·3	54·8	—	13·8	77·4	76·2	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4·9	77·7	62·6	56·2	56·8	44·8	24·4	81·4	85·9	93·8	—	—	—	—	65°·9	71°·5	42°·5	86°·5	84°
2·8	—	63·4	59·7	59·7	37·7	14·4	82·1	77·5	88·8	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	70·6	—	—	—	—	—	—	—	—	—	—	—
9·7	—	68·6	61·9	60·7	—	20·4	76·2	79·8	—	—	—	—	—	67°·3	77°·0	35°·7	86°	82°
8·4	—	57·4	51·3	52·3	—	20·8	76·0	79·3	—	—	—	—	—	—	—	—	—	—
—	72·0	60·4	56·9	56·7	—	17·2	76·5	82·9	86·8 ?	88·4 ?	—	—	—	—	—	—	—	—
9·4	—	63·6	58·3	60·2	—	20·0	86·9	—	—	—	—	—	—	70°·0	72°·9	37°·1	83°·5	81°·5
8·8	62·5	73·0	60·0	62·6	—	5·0	—	78·6	—	—	—	—	—	66°·9	73°·6	39°·5	86°·5	84°
9·4	—	60·0	56·4	57·3	41·9	13·4	84·1	84·6	93·2	93·2	—	—	—	—	—	—	—	81°·5
0·7	70·9	—	50·2	49·1	47·8	19·4	86·3	86·8	98·7	—	83·4	86·8	27·0	66°·8 ?	74°·6 ?	38°·7 ?	87°·5	83°·5
3·4	76·6	58·6	55·6	55·6	39·5	13·0	85·1	84·1	92·3 ?	—	77·6	82·3	32·1	69°·2	73°·0	37°·8	83°	81°
7·1	—	—	—	—	—	—	79·9	79·5	—	—	—	—	—	—	—	—	—	—
2·1	80·8	56·6	53·9	54·1	37·4	17·3	81·5	84·5	86·3	91·9	—	—	—	69°·0	66°·5	44°·5	80°	77°
0·9	69·9	52·4	50·8	49·9	37·6	23·8	81·5	—	86·1	90·1 ?	—	—	32·0	72°·3	68°·9	38°·8	78°	77°
—	—	—	—	—	—	32·0	82·9	79·0	91·2 ?	—	—	—	30·3	—	—	—	—	—
5·5	73·0	57·7	56·0	56·7	—	20·3	88·0	87·2	98·3	100·6	—	—	—	71°·3	66°·8	42°·0	79°·5	77°
0·6	64·6	66·1	63·8	63·0	46·7	17·9	75·3	72·1	82·6	—	74·6	80·0	35·0	70°·6	71°·7	37°·7	86°	83°·5
1·6	—	62·0	57·3	58·2	36·2	11·7	75·1	75·1	81·1	83·9	77·6	87·5	25·7	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·3	64·9	64·5	61·2	61·5	43·9	27·1	79·5	79·7	92·0	92·0	71·6	78·0	30·2	70°·8	72°·8	36°·4	82°	79°
3·6	—	57·8	54·3	54·7	40·5	20·0	79·4	79·6	85·4	87·2	77·3	84·9	29·9	—	—	—	87°·5	86 ?
8·0	—	—	—	—	—	—	—	89·0 ?	—	—	—	—	—	—	—	—	—	—
4·5	—	—	—	—	—	22·2	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·9	61·1	68·1	64·2	63·5	—	—	83·1	82·4	89·0	—	78·3	87·8	21·4	68°·7	77°·4	33°·9	85°	83°
5·5	61·2	61·8	58·1	56·9	—	19·3	85·7	81·4	94·2	—	—	—	—	65°·0	76°·1	38°·9	87°	85°·5
—	66·0	59·1	53·5	55·6	—	—	77·0	76·2	86·2	—	—	—	—	—	—	—	90°	87°·5
—	—	—	54·4	55·1	—	28·3	76·9	81·4	—	—	—	—	—	—	—	—	—	—
—	—	52·5	50·6	50·7	—	31·8	79·3	77·8	88·9	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	9·4	76·7	78·8	—	—	—	—	22·6 ?	—	—	—	—	—
7·8	—	75·0	72·0	72·5	—	13·2	81·0	81·1	—	—	—	—	—	—	—	—	—	—
8·4	—	61·6	59·1	58·1	31·3	20·5	—	76·3	—	—	82·3	87·4	27·1	—	—	—	—	—
—	—	—	—	—	—	11·7	83·9	72·4	90·5	—	—	—	—	—	—	—	—	—
1·3	62·9 ?	64·1	61·3	61·3	39·2	8·5	77·3	—	84·0	86·1	74·5	79·3	32·3	67°·5 ?	74°·2 ?	38°·3 ?	—	81°
0·5	—	65·9	62·2	62·1	39·5	19·0	—	77·3	—	—	78·6	86·7	17·6	69°·7	74°·8	35°·5	83°·5	81°
—	—	57·7	56·7	57·1	—	5·2	86·7	—	—	—	—	—	—	—	—	—	80° ?	79° ?
0·0	—	64·0	57·8	58·2	—	8·6	—	73·8	—	—	—	—	—	—	—	—	—	—
4·1	—	58·6	55·1	54·7	—	10·9	84·7	89·6	92·0	—	—	—	—	—	—	—	—	—
0·8	61·4	66·6	63·3	61·9	45·1	13·3	69·5	72·8	74·9	78·5	—	—	—	68°·1	77°·5	34°·4	86°	82°·5
2·8	—	55·3	53·0	52·6	33·6	27·5	—	82·0	—	—	—	—	—	67°·0	73°·4	39°·6	85°·5	84°
3·9	—	—	—	—	—	—	75·8 ?	—	82·3	—	—	—	—	—	—	—	—	—
4·2	62·1	63·8	56·7	58·3	34·5	—	80·2	82·9	84·9	—	77·4	86·0	28·2	71°·2	71°·0	37°·8	84°	81°
5·8	65·6	69·0	60·4	61·7	—	31·0	—	78·3	—	—	—	—	—	77°·2	65°·3	37°·5	76°·5	73°·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7·0	—	50·6	49·3	49·5	37·3 ?	26·1	79·2	78·1	87·7	91·9 ?	—	—	—	—	—	—	—	76°

Alveolar P∠	*Prosthion P∠*	θ_1	θ_2	Remarks
—	—	—	—	Teeth worn; 10 lost before death. JR.
82°	80°	26°·0	11°·1	Teeth hardly worn; 2 molars lost before death. JL.
85°	83°	26°·6	12°·9	Coronal closed, sagittal beginning to close. Teeth worn; 4 lost before death. JR.
—	—	—	—	Ageing; sagittal and coronal obliterated, lambdoid closed. Several teeth lost before death. JR.
84°·5	82°·5	32°·2	11°·1	Coronal closed, sagittal closing. Teeth worn. JR.
86°	84°·5	26°·9	12°·3	Teeth hardly worn. JR.
—	—	—	—	Ageing; coronal obliterated, sagittal nearly obliterated. Central incisors lost before death. JR.
80°	76°	—	—	Teeth hardly worn.
85°·5	85°	28°·2	10°·4	Teeth hardly worn. JR.
—	—	—	—	Coronal obliterated. Teeth worn. Incomplete face and part of calvaria only.
89°	86°	24°·7	16°·8	Teeth hardly worn. JR.
—	—	—	—	Occipital, parietals, and right temporal only.
85°	84°	29°·5	13°·1	Teeth hardly worn; 2 lost before death. JR.
—	—	—	—	Ageing; all sutures partly obliterated. J equal. Face missing.
77°·5	75°·5	33°·5	8°·6	Teeth hardly worn; left central incisor lost before death. JR.
—	—	—	—	Teeth worn; 2 lost before death. JL.
80°·5	77°·5	25°·5	10°·5	Teeth worn; 3 molars lost before death. JR.
—	—	—	—	Coronal partly obliterated. Teeth worn; some lost before death. JL.
—	—	—	—	Base and face missing.
86°·5	84°	27°·6	15°·0	Coronal closing. 11 teeth lost before death. JR.
—	—	—	—	Most teeth lost before death. JL.
—	—	—	—	Several teeth lost before death. JL.
86°	82°	26°·7	9°·0	Teeth worn; 3 lost before death. J equal.
—	—	—	—	Sagittal beginning to close. Edentulous. JR.
—	—	—	—	Aged; all sutures obliterated. Several teeth lost before death. Calvaria partly defective.
83°·5	81°·5	27°·5	10°·6	Teeth slightly worn. JR.
86°·5	84°	26°·6	12°·9	Coronal obliterated. A few teeth lost before death. JL.
—	81°·5	—	—	Teeth worn; 4 lost before death. J equal.
87°·5	83°·5	25°·7 ?	12°·9 ?	Teeth not worn. JR.
83°	81°	27°·8	10°·0	Teeth not worn. J equal.
—	—	—	—	Several teeth lost before death. JL.
80°	77°	31°·0	13°·5	Coronal closing. 4 teeth lost before death. JR.
78°	77°	29°·7	9°·1	Teeth slightly worn. JR.
—	—	—	—	Coronal and sagittal beginning to close. Teeth slightly worn. JR. Occipital bone partly defective.
79°·5	77°	29°·2	12°·7	Coronal and sagittal beginning to close. Teeth considerably worn; 1 molar lost before death. JR.
86°	83°·5	23°·4	14°·3	——
—	—	—	—	Teeth slightly worn. JR.
—	—	—	—	Teeth slightly worn; right central incisor lost before death. JR.
—	—	—	—	Base defective and face missing.
—	—	—	—	Base defective; frontal bone partly defective; face missing.
—	—	—	—	Coronal beginning to close. Base largely defective; face missing.
82°	79°	27°·2	9°·2	Frontal and parietal bones only.
87°·5	86 ?	—	—	Teeth hardly worn; 1 lost before death. J equal.
—	—	—	—	Coronal and sagittal beginning to close. Teeth worn; left central incisor lost before death. JR.
—	—	—	—	Coronal closed. Some teeth lost before death. J equal.
—	—	—	—	Coronal obliterated, sagittal closed. Face missing. J equal.
85°	83°	26°·3	7°·6	Coronal closed. Base largely defective and face missing.
87°	85°·5	28°·0	10°·9	Teeth worn. JL.
90°	87°·5	—	—	Teeth worn; 2 molars lost before death. JL.
—	—	—	—	Base defective; occipital bone missing.
—	—	—	—	Teeth hardly worn. Base defective.
—	—	—	—	Incisor lost before death. Incomplete—face and frontal bone only.
—	—	—	—	Teeth worn. Incomplete—face and frontal bone only.
—	—	—	—	7 teeth lost before death. JR.
—	—	—	—	Teeth slightly worn; 3 lost before death. JL.
—	81°	—	—	Ageing; all sutures partly obliterated. Teeth worn. Left side of calvaria and base largely defective.
83°·5	81°	26°·8	8°·7	Teeth hardly worn. JR.
80° ?	79° ?	—	—	Teeth worn; 1 lost before death. JR.
—	—	—	—	Coronal beginning to close. Teeth worn. JL.
—	—	—	—	Coronal closed on right, open on left, sagittal beginning to close. 11 teeth lost before death. JR.
86°	82°·5	25°·9	8°·5	Edentulous. JR.
85°·5	84°	27°·5	12°·1	Teeth hardly worn; 2 molars lost before death. J equal.
—	—	—	—	Teeth worn. JL.
84°	81°	24°·8	13°·0	Teeth hardly worn. J equal.
76°·5	73°·5	26°·3	11°·2	Teeth hardly worn. JL.
—	—	—	—	Coronal nearly obliterated. 9 teeth lost before death. JR. Incomplete calvaria and face.
—	76°	—	—	Coronal and sagittal nearly obliterated. 2 teeth lost before death. JR.

DISTRIBUTION OF THE MEDIAN, QUARTILES AND INTERQUARTILE DISTANCE IN SAMPLES FROM A NORMAL POPULATION

By TOKISHIGE HOJO.

Various methods of estimating from a sample the median and the standard deviation of the sampled population have been considered by K. Pearson*. If y be the ordinate of a frequency curve corresponding to a value, t, of the abscissa, and s/n the proportionate area under the curve beyond that ordinate, the sampling variation of t was expressed by him in terms of that of s by means of the equation

$$y\delta t = \delta s \quad \text{.....(1).}$$

The results so obtained were essentially for large samples, as they were based on neglecting terms in $n^{-\frac{3}{2}}$ as compared with those in $n^{-\frac{1}{2}}$.

The object of the present paper is to consider the distribution of the median, quartiles and interquartile distance (defined as described below) in the case of small samples from a normal population, and to link up the values so obtained with the limiting forms given by K. Pearson.

I. *The Median.*

Suppose that the n observations of the sample be arranged in order of magnitude, $x_1, x_2, \ldots x_n$. We shall consider only the estimates of the population median defined as follows:

(a) If the sample size be odd, and $n = 2p+1$. The median (m), is then taken as the $p+1$th value.

(b) If the sample size be even, and $n = 2p$. The median (m') is taken as the midpoint between the pth and $(p+1)$th values, or $\frac{1}{2}(x_p + x_{p+1})$.

The sampling distribution of the median will clearly be symmetrical, and expressions for the second and fourth moment coefficients only will be considered.

(1) *Limiting Values of the Moment Coefficients.*

If z_x be the ordinate of the standard normal curve

$$z_x = e^{-\frac{1}{2}x^2}/\sqrt{2\pi},$$

then K. Pearson has shown that for n odd the standard error of the median tends to

$$\sigma_m = \tilde{\sigma}/(2\sqrt{n}\, z_0) = \sqrt{\tfrac{1}{2}\pi} \times \tilde{\sigma}/\sqrt{n} = 1{\cdot}25331\tilde{\sigma}/\sqrt{n} \quad \text{.....(1·1),}$$

where $\tilde{\sigma}$ is the standard deviation in the population sampled.

* *Biometrika*, Vol. XIII. pp. 113—132.

If n be even, the limiting form may be found as follows. Let n_1 and n_2 be areas from $-\alpha$ to x_1 and x_2 respectively, $n_2 > n_1$ and $x_2 > x_1$. Then, if z_1 and z_2 be the tabular ordinates corresponding to $\frac{1}{2}(1+\alpha_1) = n_1/n$ and $\frac{1}{2}(1+\alpha_2) = n_2/n$, K. Pearson in the paper cited has shown that for n large

$$\sigma_{\frac{1}{2}(x_1+x_2)} = \frac{\tilde{\sigma}}{2\sqrt{n}} \left\{ \frac{n_1}{n}\left(1-\frac{n_1}{n}\right)\frac{1}{{z_1}^2} + \frac{n_2}{n}\left(1-\frac{n_2}{n}\right)\frac{1}{{z_2}^2} + 2\frac{n_1}{n}\left(1-\frac{n_2}{n}\right)\frac{1}{z_1 z_2} \right\}^{\frac{1}{2}}.$$

For the particular case of the median $m' = \frac{1}{2}(x_1 + x_2)$, we have

$$n_1/n = \tfrac{1}{2}\left(1-\frac{1}{n}\right), \quad n_2/n = \tfrac{1}{2}\left(1+\frac{1}{n}\right),$$

and

$$\sigma_{m'} = \frac{\tilde{\sigma}}{\sqrt{n}} \frac{1}{2z_h} \sqrt{1-\frac{1}{n}} \quad \text{.....(1·2),}$$

where z_h is to be found from $\frac{1}{2}(1+\alpha_h) = \frac{1}{2}\left(1+\frac{1}{n}\right)$.

For the case of $n = 50$, $z_h = \cdot 398{,}8169$, and

$$\sigma_{m'} = \frac{\tilde{\sigma}}{\sqrt{n}} \frac{1}{2z_h} \sqrt{1-\tfrac{1}{50}} = \frac{\tilde{\sigma}}{\sqrt{n}} \times 1{\cdot}241{,}108.$$

Had we put $n = 50$ in (1·1), $\sigma_{m'} = \dfrac{\tilde{\sigma}}{\sqrt{n}} \times 1{\cdot}253{,}314$. Thus the values of the standard deviations of the median for n even and n odd approach the same limit, but the value for n even will be less than the value for n odd. This is only a special case of K. Pearson's general proposition that the median can be found more accurately from two equally placed percentiles than from the mid-individual.

The limiting value of the momental constant β_2 may also be found using the fundamental equation (1), which enables us to express the moments of t in terms of those of the binomial. If ${}_t\mu_k$ and ${}_s\mu_k$ be the kth moment coefficients of t and s about their means, we have

$${}_t\mu_k = {}_s\mu_k / y^k \quad \text{.....(2·1),}$$

and as

$${}_s\mu_2 = n\frac{s}{n}\left(1-\frac{s}{n}\right); \quad {}_s\mu_3 = n\frac{s}{n}\left(1-\frac{s}{n}\right)\left(1-\frac{2s}{n}\right);$$

$${}_s\mu_4 = n\frac{s}{n}\left(1-\frac{s}{n}\right)\left\{1+3(n-2)\frac{s}{n}\left(1-\frac{s}{n}\right)\right\} \quad \text{.....(2·2),}$$

it follows that if we take $s/n = \frac{1}{2}$ the moment coefficients of t will be those of the median as defined above when n is odd. Or

$${}_m\beta_2 = {}_t\beta_2 = {}_t\mu_4/{}_t\mu_2{}^2 = {}_s\mu_4/{}_s\mu_2{}^2 = 3 - 2/n \quad \text{.....(3).}$$

Just as for the $(p+1)$th individual in a sample of $n = 2p+1$, $\beta_1 \to 0$ and $\beta_2 \to 3$ when $n \to \infty$, it will follow that we have the same limiting value for the pth and $(p+1)$th individuals when n is even, $= 2p$. Further, the midpoint between x_p and x_{p+1} will tend to be distributed with $\beta_1 = 0$, $\beta_2 = 3$. That is to say, whether the sample size be odd or even, the limiting form of the distribution of the median will be normal. As seen from the results below the normal form is approached very rapidly.

(2) *Theoretical Values of σ_m and ${}_m\beta_2$ in Small Samples.*

Values of σ_m for the cases $n=3$, 4 and 5 have been given by E. S. Pearson and N. K. Adyanthāya*. The method employed by them will now be extended and values for ${}_m\beta_2$ also obtained. If the n observations in a sample from a normal population with unit standard deviation and mean at zero be placed in order of magnitude, then the chance that the qth observation lies in an element $x \pm \frac{1}{2}dx$ is

$$df = \frac{n!}{(q-1)!\,(n-q)!\,1!}\,\alpha_x^{q-1}\,(1-\alpha_x)^{n-q}\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx \qquad \text{.....(4),}$$

where $$\alpha_x = \int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\,dx/\sqrt{2\pi}.$$

This is the fundamental relation upon which the following work depends.

(*a*) *Case of Sample Size odd, $n=2p+1$ ($p=1, 2, \ldots$).*

The q of (4) will now be $p+1$, and

$$\begin{aligned}
{}_n\sigma_m{}^2 &= \frac{n!}{\sqrt{2\pi}\,(p!)^2}\int_{-\infty}^{+\infty}\alpha_x{}^p\,(1-\alpha_x)^p\,x^2\,e^{-\frac{1}{2}x^2}dx \qquad \text{.....(4·1)}\\
&= \frac{n!}{(p!)^2}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\int_{-\infty}^{+\infty}\alpha_x^{p+r}\,x^2\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx\\
&= \frac{n!}{(p!)^2}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\left\{\left[-\alpha_x^{p+r}\,x\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right]_{-\infty}^{+\infty}\right.\\
&\qquad\left.+\int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\left((p+r)\,\alpha_x^{p+r-1}\,x\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}+\alpha_x^{p+r}\right)dx\right\}\\
&= \frac{n!}{(p!)^2}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\left\{\frac{p+r}{2\pi}\left(\left[-\alpha_x^{p+r-1}\,\frac{e^{-x^2}}{2}\right]_{-\infty}^{+\infty}\right.\right.\\
&\qquad\left.\left.+\frac{p+r-1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_x^{p+r-2}\,e^{-\frac{1}{2}x^2}dx\right)+\frac{1}{p+r+1}\right\}\\
&= \frac{n!}{(p!)^2}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\left\{\frac{1}{p+r+1}+\frac{(p+r)\,(p+r-1)}{4\pi}\,T_{p+r-2}\right\},
\end{aligned}$$

$${}_n\sigma_m{}^2 = 1+\frac{n!}{4\pi\,(p!)^2}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\,(p+r)\,(p+r-1)\,T_{p+r-2} \qquad \text{.....(5),}$$

where $$T_{p+r-2} = \int_{-\infty}^{+\infty}\alpha_x^{p+r-2}\,e^{-\frac{1}{2}x^2}dx/\sqrt{2\pi}.$$

Certain of these integrals have been evaluated by quadrature, and hence the values of ${}_n\sigma_m{}^2$ deduced from (5):

$$\begin{aligned}
{}_m\mu_4 &= \frac{n!}{(p!)^2}\int_{-\infty}^{+\infty}\alpha_x{}^p\,(1-\alpha_x)^p\,x^4\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx\\
&= \frac{n!}{(p!)^2}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\int_{-\infty}^{+\infty}\alpha_x^{p+r}x^4\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx.
\end{aligned}$$

* *Biometrika*, Vol. xxA. p. 359.

$$\int_{-\infty}^{+\infty} \alpha_x^{p+r} x^4 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx = \left[-x^3 \alpha_x^{p+r} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right]_{-\infty}^{+\infty}$$
$$+ \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \left((p+r)\alpha_x^{p+r-1} x^3 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + 3\alpha_x^{p+r} x^2\right) dx$$
$$= \frac{p+r}{2\pi} \left\{\left[-\alpha_x^{p+r-1} x^2 \frac{e^{-x^2}}{2}\right]_{-\infty}^{+\infty}\right.$$
$$\left. + \tfrac{1}{2} \int_{-\infty}^{+\infty} e^{-x^2} \left((p+r-1)\alpha_x^{p+r-2} x^2 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + 2\alpha_x^{p+r-1} x\right) dx\right\}$$
$$+ 3\left\{\left[-\alpha_x^{p+r} x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \left((p+r)\alpha_x^{p+r-1} x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + \alpha_x^{p+r}\right) dx\right\}$$
$$= \frac{(p+r)(p+r-1)}{4\pi\sqrt{2\pi}} \left\{\left[-\tfrac{1}{3}\alpha_x^{p+r-2} x e^{-\frac{3}{2}x^2}\right]_{-\infty}^{+\infty}\right.$$
$$\left. + \tfrac{1}{3} \int_{-\infty}^{+\infty} e^{-\frac{3}{2}x^2} \left((p+r-2)\alpha_x^{p+r-3} x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + \alpha_x^{p+r-2}\right) dx\right\}$$
$$+ \frac{4(p+r)}{2\pi} \left\{\left[-\alpha_x^{p+r-1} \frac{e^{-x^2}}{2}\right]_{-\infty}^{+\infty} + \frac{p+r-1}{2} \int_{-\infty}^{+\infty} \alpha_x^{p+r-2} \frac{e^{-\frac{3}{2}x^2}}{\sqrt{2\pi}}\,dx\right\} + \frac{3}{p+r+1}$$
$$= \frac{(p+r)(p+r-1)(p+r-2)}{24\pi^2} \left\{\left[-\tfrac{1}{4}\alpha_x^{p+r-3} e^{-2x^2}\right]_{-\infty}^{+\infty}\right.$$
$$\left. + \frac{p+r-3}{4} \int_{-\infty}^{+\infty} \alpha_x^{p+r-4} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx\right\}$$
$$+ \frac{13(p+r)(p+r-1)}{12\pi} T_{p+r-2} + \frac{3}{p+r+1}$$
$$= \frac{3}{p+r+1} + \frac{13(p+r)(p+r-1)}{12\pi} T_{p+r-2}$$
$$+ \frac{(p+r)(p+r-1)(p+r-2)(p+r-3)}{96\pi^2} S_{p+r-4},$$

where
$$S_{p+r-4} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha_x^{p+r-4} e^{-\frac{1}{2}x^2}\,dx,$$

$$_m\mu_4 = 3 + \frac{n!}{12\pi(p!)^2} \sum_{r=0}^{p} (-1)^r {}_pC_r (p+r)(p+r-1) \left\{13T_{p+r-2}\right.$$
$$\left. + \frac{(p+r-2)(p+r-3)}{8\pi} S_{p+r-4}\right\} \quad \ldots\ldots\ldots(6).$$

Certain of the integrals are tabled below.

(*b*) *Case of Sample Size even, $n = 2p$.*

Writing for simplicity x_1 and x_2 for the pth and $(p+1)$th observations we can find the standard deviation from the following formula:

$$\sigma_{m'}^2 = \sigma_{\frac{1}{2}(x_1+x_2)}^2 = \tfrac{1}{4}(x_1\mu_2 + 2p_{11} + x_1\mu_2) = \tfrac{1}{2}(x_1\mu_2 + p_{11}) \ldots\ldots\ldots\ldots(7),$$

where p_{11} is the mean value of the product x_1x_2, and ${}_{x_1}\mu_2 = {}_{x_2}\mu_2$, since the population is symmetrical. Further,

$$p_{11} = \frac{n!}{\{(p-1)!\}^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1} \alpha_{x_2}^{p-1}(1-\alpha_{x_1})^{p-1}\, x_1x_2\, \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\, dx_2dx_1$$

$$= \frac{n!}{\{(p-1)!\}^2}\sum_{r=0}^{p-1}(-1)^r\, {}_{p-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^r\alpha_{x_2}^{p-1}\, x_1x_2\, \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\, dx_2dx_1.$$

Now

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^r\alpha_{x_2}^{p-1}\, x_1x_2\, \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\, dx_2dx_1$$

$$= \int_{-\infty}^{+\infty}\alpha_{x_1}^r x_1\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_1\left\{\left[-\alpha_{x_2}^{p-1}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\right]_{-\infty}^{x_1} + \frac{p-1}{2\pi}\int_{-\infty}^{x_1}\alpha_{x_2}^{p-2}\, e^{-x_2^2}dx_2\right\}$$

$$= -\frac{1}{2\pi}\left\{\left[-\alpha_{x_1}^{p+r-1}\frac{e^{-x_1^2}}{2}\right]_{-\infty}^{+\infty} + \frac{p+r-1}{2}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r-2}\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_1\right\}$$

$$+ \frac{p-1}{(\sqrt{2\pi})^3}\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1}^r\alpha_{x_1-x_2'}^{p-2}\, x_1 e^{-\frac{1}{2}(x_1^2+2(x_1-x_2')^2)}\, dx_2'\, dx_1$$

$$= -\frac{p+r-1}{4\pi}T_{p+r-2} + \frac{p-1}{(\sqrt{2\pi})^3}\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1'}^{p-2}\alpha_{x_1'+x_2'}^r(x_1'+x_2')$$

$$\times e^{-\frac{1}{2}((x_1'+x_2')^2+2x_1'^2)}dx_2'\, dx_1'$$

$$= -\frac{p+r-1}{4\pi}T_{p+r-2} + \frac{p-1}{(\sqrt{2\pi})^3}\int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-2}e^{-x_1'^2}\left\{\left[-\alpha_{x_1'+x_2'}^r e^{-\frac{1}{2}(x_1'+x_2')^2}\right]_0^{\infty}\right.$$

$$\left. + r\int_0^{\infty}\alpha_{x_1'+x_2'}^{r-1}\frac{e^{-(x_1'+x_2')^2}}{\sqrt{2\pi}}dx_2'\right\}dx_1'$$

$$= -\frac{p+r-1}{4\pi}T_{p+r-2} + \frac{p-1}{2\pi}\left\{T_{p+r-2} + \frac{r}{2\pi}\int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-2}e^{-x_1'^2}dx_1'\right.$$

$$\left.\times\left(\sqrt{2\pi}\, I_{r-1} - \int_{-\infty}^{x_1'}\alpha_{x_2''}^{r-1}e^{-x_2''^2}dx_2''\right)\right\}$$

$$= \frac{p-r-1}{4\pi}T_{p+r-2} + \frac{(p-1)r}{2\pi}(I_{p-2}I_{r-1} - {}_{p-2}I_{r-1}),$$

where
$$I_{r-1} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_x^{r-1}e^{-x^2}dx$$

and
$${}_{p-2}I_{r-1} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p-2}e^{-x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{r-1}e^{-x_2^2}dx_2.$$

Hence

$$p_{11} = \frac{n!}{4\pi\{(p-1)!\}^2}\sum_{r=0}^{p-1}(-1)^r\, {}_{p-1}C_r\{(p-r-1)\, T_{p+r-2} + 2(p-1)\, r_{r-1}I_{p-2}\} \qquad \text{.....(8).}$$

Again

$${}_{x_1}\mu_2 = \frac{n!}{\{(p-1)!\}^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_2}^{p-1}(1-\alpha_{x_1})^{p-1}x_1^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2dx_1$$

$$= \frac{n!}{\{(p-1)!\}^2}\sum_{r=0}^{p-1}(-1)^r\, {}_{p-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^r\alpha_{x_2}^{p-1}x_1^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2dx_1$$

$$x_1\mu_2 = \frac{n!}{p!(p-1)!}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r}x_1^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}dx_1$$

$$= \frac{n!}{p!(p-1)!}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\left\{\frac{1}{p+r+1}+\frac{(p+r)(p+r-1)}{4\pi}T_{p+r-2}\right\}$$

$$= 1+\frac{n!}{4\pi p!(p-1)!}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r(p+r)(p+r-1)\,T_{p+r-2} \quad \ldots(9).$$

On substituting (8) and (9) in (7) we obtain

$$\sigma^2_{m'} = \sigma^2_{\frac{1}{2}(x_1+x_2)} = \tfrac{1}{2}+\frac{n!}{8\pi(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r$$

$$\times\left\{\frac{2p(p-1)+(p+r-1)r}{p}T_{p+r-2}+2(p-1)\,r_{r-1}I_{p-2}\right\}\ldots\ldots(10).$$

The fourth moment coefficient can be found from the following formula:

$$m'\mu_4 = \tfrac{1}{4}(x_1+x_2)\mu_4 = \tfrac{1}{16}(x_4\mu_4+4p_{31}+6p_{22}+4p_{13}+x_4\mu_4) = \tfrac{1}{8}(x_4\mu_4+4p_{31}+3p_{22})$$

$\ldots\ldots(11)$,

where p_{rs} is the mean value of the product $x_1^r x_2^s$, since in our case

$$x_3\mu_4 = x_1\mu_4, \quad p_{13} = p_{31},$$

$$x_1\mu_4 = \frac{n!}{(p-1)!^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{p-1}(1-\alpha_{x_1})^{p-1}x_1^4\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2\,dx_1$$

$$= \frac{n!}{(p-1)!^2\,p}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r}x_1^4\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}dx_1$$

$$= \frac{n!}{p!(p-1)!}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\left\{\frac{3}{p+r+1}+\frac{13(p+r)(p+r-1)}{12\pi}T_{p+r-2}\right.$$

$$\left.+\frac{(p+r)(p+r-1)(p+r-2)(p+r-3)}{96\pi^2}S_{p+r-4}\right\}$$

$$= 3+\frac{n!}{12\pi p!(p-1)!}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r(p+r)(p+r-1)\left\{13T_{p+r-2}\right.$$

$$\left.+\frac{(p+r-2)(p+r-3)}{8\pi}S_{p+r-4}\right\}\ldots\ldots(12),$$

$$p_{31} = \frac{n!}{(p-1)!^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{p-1}(1-\alpha_{x_1})^{p-1}x_1^3x_2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2\,dx_1$$

$$= \frac{n!}{(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^r\alpha_{x_2}^{p-1}x_1^3x_2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2\,dx_1$$

$$= \frac{n!}{(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^r x_1^3\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}dx_1$$

$$\times\left(-\alpha_{x_1}^{p-1}\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}+\frac{p-1}{2\pi}\int_{-\infty}^{x_1}\alpha_{x_2}^{p-2}e^{-x_2^2}dx_2\right)$$

$$= \frac{n!}{2\pi(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\left\{-\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r-1}x_1^3e^{-x_1^2}dx_1\right.$$

$$\left.+\frac{p-1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^r\alpha_{x_2}^{p-2}x_1^3e^{-\frac{1}{2}(x_1^2+2x_2^2)}dx_2\,dx_1\right\}\ldots\ldots(13{\cdot}1),$$

$$\int_{-\infty}^{+\infty} a_x{}^r x^3 e^{-x^2} dx = \left[-a_x{}^r x^2 \frac{e^{-x^2}}{2} \right]_{-\infty}^{+\infty} + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-x^2} \left(r a_x^{r-1} x^2 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + 2 a_x{}^r x \right) dx$$

$$= \frac{r}{2\sqrt{2\pi}} \left\{ \left[-\tfrac{1}{3} a_x^{r-1} x e^{-\frac{3}{2}x^2} \right]_{-\infty}^{+\infty} + \tfrac{1}{3} \int_{-\infty}^{+\infty} e^{-\frac{3}{2}x^2} \left((r-1) a_x^{r-2} x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + a_x^{r-1} \right) dx \right\}$$

$$+ \left[-a_x{}^r \frac{e^{-x^2}}{2} \right]_{-\infty}^{+\infty} + \frac{r}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} a_x^{r-1} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{r(r-1)}{12\pi} \left\{ \left[-\tfrac{1}{4} a_x^{r-2} e^{-2x^2} \right]_{-\infty}^{+\infty} + \frac{r-2}{4} \int_{-\infty}^{+\infty} a_x^{r-3} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \right\} + \frac{r}{6} T_{r-1} + \frac{r}{2} T_{r-1}$$

$$= \frac{2r}{3} T_{r-1} + \frac{r(r-1)(r-2)}{48\pi} S_{r-3} \quad \text{.....(13·2),}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} a_{x_1}^r a_{x_2}^{p-2} x_1{}^3 e^{-\frac{1}{2}(x_1^2+2x_2^2)} dx_2 dx_1 = \int_{-\infty}^{+\infty} \int_0^{\infty} a_{x_1}^r a_{x_1-x_2'}^{p-2} x_1{}^3 e^{-\frac{1}{2}(x_1^2+2(x_1-x_2')^2)} dx_2' dx_1$$

$$= \int_{-\infty}^{+\infty} a_{x_1'}^{p-2} e^{-x_1'^2} dx_1' \int_0^{\infty} a_{x_1'+x_2'}^r (x_1'+x_2')^3 e^{-\frac{1}{2}(x_1'+x_2')^2} dx_2' \text{ ...(13·3),}$$

$$\int_0^{\infty} a_{x_1+x_2}^r (x_1+x_2)^3 e^{-\frac{1}{2}(x_1+x_2)^2} dx_2 = \left[-a_{x_1+x_2}^r (x_1+x_2)^2 e^{-\frac{1}{2}(x_1+x_2)^2} \right]_{x_2=0}^{\infty}$$

$$+ \int_0^{\infty} e^{-\frac{1}{2}(x_1+x_2)^2} \left\{ r a_{x_1+x_2}^{r-1} (x_1+x_2)^2 \frac{e^{-\frac{1}{2}(x_1+x_2)^2}}{\sqrt{2\pi}} \right.$$

$$\left. + 2(x_1+x_2) a_{x_1+x_2}^r \right\} dx_2 = a_{x_1}^r x_1{}^2 e^{-\frac{1}{2}x_1^2} + \frac{r}{\sqrt{2\pi}} \left\{ \left[-a_{x_1+x_2}^{r-1} \tfrac{1}{2} (x_1+x_2) e^{-(x_1+x_2)^2} \right]_0^{\infty} \right.$$

$$\left. + \tfrac{1}{2} \int_0^{\infty} e^{-(x_1+x_2)^2} \left[(r-1) a_{x_1+x_2}^{r-2} (x_1+x_2) \frac{e^{-\frac{1}{2}(x_1+x_2)^2}}{\sqrt{2\pi}} + a_{x_1+x_2}^{r-1} \right] dx_2 \right\}$$

$$+ 2 \left\{ \left[-a_{x_1+x_2}^r e^{-\frac{1}{2}(x_1+x_2)^2} \right]_0^{\infty} + r \int_0^{\infty} a_{x_1+x_2}^{r-1} \frac{e^{-(x_1+x_2)^2}}{\sqrt{2\pi}} dx_2 \right\}$$

$$= a_{x_1}^r x_1{}^2 e^{-\frac{1}{2}x_1^2} + \frac{r}{2\sqrt{2\pi}} a_{x_1}^{r-1} x_1 e^{-x_1^2} + 2 a_{x_1}^r e^{-\frac{1}{2}x_1^2}$$

$$+ \frac{r}{\sqrt{2\pi}} (\tfrac{1}{2} + 2) \int_0^{\infty} a_{x_1+x_2}^{r-1} e^{-(x_1+x_2)^2} dx_2$$

$$+ \frac{r(r-1)}{4\pi} \left\{ \left[-\tfrac{1}{3} a_{x_1+x_2}^{r-2} e^{-\frac{3}{2}(x_1+x_2)^2} \right]_0^{\infty} + \frac{r-2}{3} \int_0^{\infty} a_{x_1+x_2}^{r-3} \frac{e^{-2(x_1+x_2)^2}}{\sqrt{2\pi}} dx_2 \right\}.$$

On substituting in (13·3), we obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} a_{x_1}^r a_{x_2}^{p-2} x_1{}^3 e^{-\frac{1}{2}(x_1^2+2x_2^2)} dx_2 dx_1 = \int_{-\infty}^{+\infty} a_{x_1'}^{p+r-2} x_1'^2 e^{-\frac{3}{2}x_1'^2} dx_1'$$

$$+ \frac{r}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} a_{x_1'}^{p+r-3} x_1' e^{-2x_1'^2} dx_1'$$

$$+ 2\sqrt{2\pi} T_{p+r-2} + \frac{r(r-1)}{6\sqrt{2\pi}} S_{p+r-4} + \frac{5r\sqrt{2\pi}}{2} {}_{r-1}I_{p-2} + \frac{r(r-1)(r-2)}{6\sqrt{2\pi}} {}_{r-3}K_{p-2}$$

.....(13·4),

where $$_{r-3}K_{p-2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{0}^{\infty} a_{x_1}^{p-2} a_{x_1+x_2}^{r-3} e^{-x_1^2-2(x_1+x_2)^2}\,dx_2\,dx_1$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} a_{x_1}^{r-3} e^{-2x_1^2}\,dx_1 \int_{-\infty}^{x_1} a_{x_2}^{p-2} e^{-x_2^2}\,dx_2,$$

$$\int_{-\infty}^{+\infty} a_x^{p+r-2} x^2 e^{-\frac{1}{2}x^2}\,dx = \left[-\tfrac{1}{3} a_x^{p+r-2} x e^{-\frac{1}{2}x^2}\right]_{-\infty}^{+\infty}$$

$$+ \tfrac{1}{3}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\left((p+r-2)a_x^{p+r-3} x\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + a_x^{p+r-2}\right)dx$$

$$= \frac{p+r-2}{3\sqrt{2\pi}}\left\{\left[-\tfrac{1}{4} a_x^{p+r-3} e^{-2x^2}\right]_{-\infty}^{+\infty} + \frac{p+r-3}{4}\int_{-\infty}^{+\infty} a_x^{p+r-4}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx\right\} + \frac{\sqrt{2\pi}}{3} T_{p+r-2}$$

$$= \frac{(p+r-2)(p+r-3)}{12\sqrt{2\pi}} S_{p+r-4} + \frac{\sqrt{2\pi}}{3} T_{p+r-2} \quad \text{.....(13·5),}$$

and then

$$\int_{-\infty}^{+\infty} a_x^{p+r-3} x e^{-2x^2}\,dx = \left[-\tfrac{1}{4} a_x^{p+r-3} e^{-2x^2}\right]_{-\infty}^{+\infty} + \frac{p+r-3}{4}\int_{-\infty}^{+\infty} a_x^{p+r-4}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx$$

$$= \frac{p+r-3}{4} S_{p+r-4} \quad \text{.....(13·6).}$$

On substituting (13·5) and (13·6) in (13·4), we find

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1} a_{x_1}^{r} a_{x_2}^{p-2} x_1^3 e^{-\frac{1}{2}(x_1^2+2x_2^2)}\,dx_2\,dx_1 = \frac{7\sqrt{2\pi}}{3} T_{p+r-2}$$

$$+ \frac{1}{2\sqrt{2\pi}}\left\{\frac{(p+r-2)(p+r-3)}{6} + \frac{r(r-1)}{3} + \frac{r(p+r-3)}{4}\right\} S_{p+r-4}$$

$$+ \frac{5r\sqrt{2\pi}}{2}\,{}_{r-1}I_{p-2} + \frac{r(r-1)(r-2)}{6\sqrt{2\pi}}\,{}_{r-3}K_{p-2} \quad \text{...(13·7).}$$

On substituting (13·2) and (13·7) in (13·1), we obtain

$$p_{31} = \frac{n!}{2\pi(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\left[\left\{\frac{7(p-1)}{3} - \frac{2(p+r-1)}{3}\right\} T_{p+r-2}\right.$$

$$+ \frac{1}{4\pi}\{\tfrac{1}{12}(2p-2-p-r+1)(p+r-2)(p+r-3)$$

$$\left. + \tfrac{1}{3}r(r-1)(p-1) + \tfrac{1}{4}r(p-1)(p+r-3)\right\} S_{p+r-4} + \frac{5r(p-1)}{2}\,{}_{r-1}I_{p-2}$$

$$\left. + \frac{(p-1)r(r-1)(r-2)}{12\pi}\,{}_{r-3}K_{p-2}\right],$$

or, finally:

$$p_{31} = \frac{n!}{2\pi(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r\,{}_{p-1}C_r\left[\frac{5(p-1)-2r}{3} T_{p+r-2}\right.$$

$$+ \frac{1}{48\pi}\{(p-r-1)(p+r-2)(p+r-3) + 4(p-1)r(r-1)$$

$$\left. + 3(p-1)r(p+r-3)\right\} S_{p+r-4} + \frac{5r(p-1)}{2}\,{}_{r-1}I_{p-2}$$

$$\left. + \frac{(p-1)r(r-1)(r-2)}{12\pi}\,{}_{r-3}K_{p-2}\right] \quad \text{.....(13),}$$

$$p_{22}=\frac{n!}{(p-1)!^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_2}^{p-1}(1-\alpha_{x_1})^{p-1}x_1^2x_2^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2dx_1$$

$$=\frac{n!}{(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r{}_{p-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^r\alpha_{x_2}^{p-1}x_1^2x_2^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2dx_1$$

$$=\frac{n!}{(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r{}_{p-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^r x_1^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}dx_1\left\{\left[-\alpha_{x_2}^{p-1}x_2\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\right]_{-\infty}^{x_1}\right.$$
$$\left.+\int_{-\infty}^{x_1}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\left((p-1)\alpha_{x_2}^{p-2}x_2\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}+\alpha_{x_2}^{p-1}\right)dx_2\right\}$$

$$=\frac{n!}{(p-1)!}\sum_{r=0}^{p-1}(-1)^r{}_{p-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^r x_1^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}dx_1\left\{-\alpha_{x_1}^{p-1}x_1\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}+\frac{\alpha_{x_1}^p}{p}\right.$$
$$\left.+\frac{p-1}{2\pi}\left(\left[-\alpha_{x_2}^{p-2}\frac{e^{-x_2^2}}{2}\right]_{-\infty}^{x_1}+\frac{1}{2}(p-2)\int_{-\infty}^{x_1}\alpha_{x_2}^{p-3}\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}dx_2\right)\right\}$$

$$=\frac{n!}{(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r{}_{p-1}C_r\left\{\frac{1}{p}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r}x_1^2\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}dx_1\right.$$
$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r-1}x_1^3e^{-x_1^2}e^{-x_1^2}dx_1-\frac{p-1}{4\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r-2}x_1^2e^{-\frac{1}{2}x_1^2}dx_1$$
$$\left.+\frac{(p-1)(p-2)}{8\pi^2}\int_{-\infty}^{+\infty}\alpha_{x_1}^r x_1^2e^{-\frac{1}{2}x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{p-3}e^{-\frac{1}{2}x_2^2}dx_2\right\}\quad\text{.....(14·1).}$$

But

$$\int_{-\infty}^{+\infty}\alpha_x^{p+r}x^2\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx=\frac{1}{p+r+1}+\frac{(p+r)(p+r-1)}{4\pi}T_{p+r-2}\quad\text{.....(14·2),}$$

$$\int_{-\infty}^{+\infty}\alpha_x^{p+r-1}x^3e^{-x^2}dx=\frac{2(p+r-1)}{3}T_{p+r-2}+\frac{(p+r-1)(p+r-2)(p+r-3)}{48\pi}S_{p+r-4}$$
$$\text{.....(14·3),}$$

$$\int_{-\infty}^{+\infty}\alpha_x^{p+r-2}x^2e^{-\frac{1}{2}x^2}dx=\frac{(p+r-2)(p+r-3)}{12\sqrt{2\pi}}S_{p+r-4}+\frac{\sqrt{2\pi}}{3}T_{p+r-2}\quad\text{.....(14·4).}$$

While

$$\int_{-\infty}^{+\infty}\alpha_{x_1}^r x_1^2e^{-\frac{1}{2}x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{p-3}e^{-\frac{1}{2}x_2^2}dx_2$$
$$=\int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-3}e^{-\frac{1}{2}x_1'^2}dx_1'\int_0^{\infty}\alpha_{x_1'+x_2'}^r(x_1'+x_2')^2e^{-\frac{1}{2}(x_1'+x_2')^2}dx_2'$$

$$=\int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-3}e^{-\frac{1}{2}x_1'^2}dx_1'\left\{\left[-\alpha_{x_1'+x_2'}^r(x_1'+x_2')e^{-\frac{1}{2}(x_1'+x_2')^2}\right]_{x_2'=0}^{\infty}\right.$$
$$\left.+\int_0^{\infty}e^{-\frac{1}{2}(x_1'+x_2')^2}\left(r\alpha_{x_1'+x_2'}^{r-1}(x_1'+x_2')\frac{e^{-\frac{1}{2}(x_1'+x_2')^2}}{\sqrt{2\pi}}+\alpha_{x_1'+x_2'}^r\right)dx_2'\right\}$$

$$=\left[-\frac{1}{4}\alpha_{x_1'}^{p+r-3}e^{-2x_1'^2}\right]_{-\infty}^{+\infty}+\frac{1}{4}\int_{-\infty}^{+\infty}(p+r-3)\alpha_{x_1'}^{p+r-4}\frac{e^{-\frac{1}{2}x_1'^2}}{\sqrt{2\pi}}dx_1+$$ (see p. 324)

$$+\frac{r}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_{x_1'}^{p-3}e^{-\frac{1}{2}x_1'^2}dx_1'\left\{\left[-a_{x_1'+x_2'}^{r-1}\tfrac{1}{2}e^{-(x_1'+x_2')^2}\right]_0^\infty\right.$$

$$\left.+\int_0^\infty\frac{r-1}{2}a_{x_1'+x_2'}^{r-2}\frac{e^{-\frac{1}{2}(x_1'+x_2')^2}}{\sqrt{2\pi}}dx_2'\right\}$$

$$+\frac{2\pi}{r+1}(T_{p-3}-T_{p+r-2})$$

$$=\tfrac{1}{2}\left(r+\frac{p+r-3}{2}\right)S_{p+r-4}+\frac{2\pi}{r+1}(T_{p-3}-T_{p+r-2})+\frac{r(r-1)}{2}{}_{r-2}T_{p-3}$$

.....(14·5),

where $${}_{r-2}T_{p-3}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1'}^{p-3}e^{-\frac{1}{2}x_1'^2}dx_1'\int_0^\infty a_{x_1'+x_2'}^{r-2}e^{-\frac{1}{2}(x_1'+x_2')^2}$$

$$=T_{p-3}T_{r-2}-\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}^{p-3}e^{-\frac{1}{2}x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}^{r-2}e^{-\frac{1}{2}x_2^2}dx_2.$$

On substituting (14·2), (14·3), (14·4) and (14·5) in (14·1), we obtain

$$p_{22}=\frac{n!}{(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r{}_{p-1}\left\{\frac{1}{p(p+r+1)}+\frac{(p+r)(p+r-1)}{4\pi p}T_{p+r-2}\right.$$

$$-\frac{p+r-1}{3\pi}T_{p+r-2}-\frac{(p+r-1)(p+r-2)(p+r-3)}{96\pi^2}S_{p+r-4}$$

$$-\frac{(p-1)(p+r-2)(p+r-3)}{96\pi^2}S_{p+r-4}-\frac{p-1}{12\pi}T_{p+r-2}$$

$$+\frac{(p-1)(p-2)(p+3r-3)}{32\pi^2}S_{p+r-4}$$

$$\left.+\frac{(p-1)(p-2)}{4\pi(r+1)}(T_{p-3}-T_{p+r-2})+\frac{(p-1)(p-2)r(r-1)}{16\pi^2}{}_{r-2}T_{p-3}\right\}$$

$$=1+\frac{n!}{4\pi p!(p-3)!}T_{p-3}+\frac{n!}{4\pi(p-1)!^2}\sum_{r=0}^{p-1}(-1)^r{}_{p-1}C_r\left\{\left[\frac{(p+r)(p+r-1)}{p}\right.\right.$$

$$\left.-\frac{5(p-1)+4r}{3}-\frac{(p-1)(p-2)}{r+1}\right]T_{p+r-2}$$

$$+\frac{1}{24\pi}[3(p-1)(p-2)(p+3r-3)-(2p+r-2)(p+r-2)(p+r-3)]$$

$$\left.\times S_{p+r-4}+\frac{(p-1)(p-2)r(r-1)}{4\pi}{}_{r-2}T_{p-3}\right\}$$ .....(14).

By substituting (12), (13) and (14) in (11), we obtain μ_4 and then we can find β_2.

It was next necessary to calculate the integral forms T_r, I_r, ${}_pI_r$, S_r, ${}_pK_r$ and ${}_pT_r$ to obtain numerical values for σ_m and ${}_m\beta_2$. The results, together with those for certain other integrals required below, were obtained by quadrature and are given in Table I; the values marked with an asterisk were previously found to seven decimal places by E. S. Pearson*.

* *Biometrika*, Vol. XVIII. p. 178.

TABLE I.

$$T_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,2}\ e^{-\frac{1}{2}x^2}dx = \cdot 1675\ 5590^{*} \qquad I_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,2}\ e^{-x^2}dx = \cdot 2150\ 2173^{*}$$

$$T_3 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,3}\ e^{-\frac{1}{2}x^2}dx = \cdot 1069\ 9628^{*} \qquad I_3 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,3}\ e^{-x^2}dx = \cdot 1457\ 5597^{*}$$

$$T_4 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,4}\ e^{-\frac{1}{2}x^2}dx = \cdot 0730\ 2284 \qquad I_4 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,4}\ e^{-x^2}dx = \cdot 1058\ 8050^{*}$$

$$T_5 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,5}\ e^{-\frac{1}{2}x^2}dx = \cdot 0523\ 4250 \qquad I_5 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,5}\ e^{-x^2}dx = \cdot 0807\ 0019$$

$$T_6 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,6}\ e^{-\frac{1}{2}x^2}dx = \cdot 0389\ 6118 \qquad I_6 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,6}\ e^{-x^2}dx = \cdot 0637\ 2205$$

$$T_7 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,7}\ e^{-\frac{1}{2}x^2}dx = \cdot 0298\ 8186 \qquad I_7 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,7}\ e^{-x^2}dx = \cdot 0516\ 9967$$

$$T_8 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,8}\ e^{-\frac{1}{2}x^2}dx = \cdot 0234\ 8297 \qquad I_8 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,8}\ e^{-x^2}dx = \cdot 0428\ 5646$$

$$T_9 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,9}\ e^{-\frac{1}{2}x^2}dx = \cdot 0188\ 3072 \qquad I_9 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,9}\ e^{-x^2}dx = \cdot 0361\ 5095$$

$$T_{10} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,10} e^{-\frac{1}{2}x^2}dx = \cdot 0153\ 5937 \qquad I_{10} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,10} e^{-x^2}dx = \cdot 0309\ 3838$$

$$I_{11} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,11} e^{-x^2}dx = \cdot 0268\ 0146$$

$$R_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x\ a_{\sqrt{2}x} e^{-x^2}dx = \cdot 2241\ 0342 \qquad R_5 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,5} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 0686\ 6630$$

$$R_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,2} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 1548\ 3762 \qquad R_6 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,6} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 0557\ 0023$$

$$R_3 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,3} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 1134\ 9591 \qquad R_7 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,7} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 0461\ 2510$$

$$R_4 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,4} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 0868\ 4937 \qquad R_8 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a_x^{\,8} a_{\sqrt{2}x} e^{-x^2}dx = \cdot 0388\ 4985$$

$${}_2I_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} a_{x_1}{}^2 e^{-x_1^2}dx_1 \int_{-\infty}^{x_1} a_{x_2}\ e^{-x_2^2}dx_2 = \cdot 0453\ 6105$$

$${}_3I_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} a_{x_1}{}^3 e^{-x_1^2}dx_1 \int_{-\infty}^{x_1} a_{x_2}\ e^{-x_2^2}dx_2 = \cdot 0344\ 1965$$

$${}_4I_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} a_{x_1}{}^4 e^{-x_1^2}dx_1 \int_{-\infty}^{x_1} a_{x_2}\ e^{-x_2^2}dx_2 = \cdot 0270\ 1597$$

$${}_5I_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} a_{x_1}{}^5 e^{-x_1^2}dx_1 \int_{-\infty}^{x_1} a_{x_2}\ e^{-x_2^2}dx_2 = \cdot 0217\ 7567$$

TABLE I—(*continued*).

$${}_6I_1=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^6e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}\,e^{-x_2^2}dx_2=\cdot0179\,3148$$

$${}_7I_1=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^7e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}\,e^{-x_2^2}dx_2=\cdot0150\,2805$$

$${}_8I_1=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^8e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}\,e^{-x_2^2}dx_2=\cdot0127\,8147$$

$${}_3I_2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^3e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}{}^2e^{-x_2^2}dx_2=\cdot0180\,4335$$

$${}_4I_2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^4e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}{}^2e^{-x_2^2}dx_2=\cdot0144\,7149$$

$${}_5I_2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^5e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}{}^2e^{-x_2^2}dx_2=\cdot0118\,6405$$

$${}_6I_2=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^6e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}{}^2e^{-x_2^2}dx_2=\cdot0099\,0350$$

$${}_4I_3=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^4e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}{}^3e^{-x_2^2}dx_2=\cdot0086\,7708$$

$${}_5I_3=\frac{1}{2\pi}\int_{-\infty}^{+\infty}a_{x_1}{}^5e^{-x_1^2}dx_1\int_{-\infty}^{x_1}a_{x_2}{}^3e^{-x_2^2}dx_2=\cdot0072\,1947$$

$$S_2=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^2e^{-\frac{1}{2}x^2}dx=\cdot0196\,9093 \qquad K_2=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^2e^{-2x^2}dx=\cdot1410\,2355^*$$

$$S_3=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^3e^{-\frac{1}{2}x^2}dx=\cdot0117\,4232 \qquad K_3=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^3e^{-2x^2}dx=\cdot0865\,3532$$

$$S_4=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^4e^{-\frac{1}{2}x^2}dx=\cdot0074\,1828 \qquad K_4=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^4e^{-2x^2}dx=\cdot0565\,0627$$

$$S_5=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^5e^{-\frac{1}{2}x^2}dx=\cdot0049\,0651 \qquad K_5=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^5e^{-2x^2}dx=\cdot0387\,0679$$

$$K_6=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^6e^{-2x^2}dx=\cdot0275\,4161$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x\,a_{\sqrt{3}x}e^{-\frac{1}{2}x^2}dx=\cdot1775\,4295 \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^3a_{\sqrt{2}x}e^{-2x^2}dx=\cdot0603\,8166$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^2a_{\sqrt{2}x}e^{-x^2}dx=\cdot1627\,5459 \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^3a_{\sqrt{2}x}e^{-\frac{1}{2}x^2}dx=\cdot0782\,5593$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^2a_{\sqrt{2}x}e^{-2x^2}dx=\cdot0912\,9402 \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}a_x{}^4a_{\sqrt{2}x}e^{-\frac{1}{2}x^2}dx=\cdot0564\,6192$$

Substituting these numerical values in the preceding equations we can obtain the theoretical values of σ_m and $\sigma_{m'}$ for $n=1, 2, \ldots 12$, and of ${}_m\beta_2$ and ${}_{m'}\beta_2$ for $n=1, 2, \ldots 8$. The results are given in Table II[a].

TABLE II[a].

Theoretical Moment Coefficients of Median.

Size of Sample, n	Standard Deviation	μ_2	μ_4	β_2	
1	1·000 00	1·000 0000	3·000 000	3·0000	
2	·707 11	·500 0000	·750 000		3·0000
3	·669 83	·448 6712	·610 908	3·0347	
4	·546 08	·298 2004	·268 471		3·0191
5	·535 57	·286 8341	·249 456	3·0320	
6	·463 40	·214 7416	·139 319		3·0212
7	·458 74	·210 4459	·134 065	3·0272	
8	·410 11	·168 1880	·085 436		3·0200
9	·407 55	·166 0931	—	—	
10	·372 14	·138 4887	—		—
11	·370 44	·137 2266	—	—	
12	·343 46	·117 9623	—		—

It will be remembered that the population standard deviation has been taken as unity. If we write

Standard error of median $= c_m \times \tilde{\sigma}/\sqrt{n} = c_m \times$ Standard error of mean,

we obtain values for c_m shown in Table II[b]. There are seen to be two series of values, c_m and $c_{m'}$, according as n be odd or even, which tend separately towards the limiting

TABLE II[b].

Values of c_m and $c_{m'}$.

n	c_m	$c_{m'}$	n	c_m	$c_{m'}$
1	1·0000		8		1·1600
2		1·0000	9	1·2226	
3	1·1602		10		1·1768
4		1·0922	11	1·2286	
5	1·1976		12		1·1898
6		1·1351			
7	1·2137		large	1·2533	1·2533

value of 1·2533. For n even the convergence is the slower. For practical guidance it appeared useful to fill in the gaps between $n = 11$ and ∞, and $n = 12$ and ∞, with two series of approximate values. Equations of the form

$$y = a_0 + a_1/n + a_2/n^2 + \ldots$$

were used, a_0 being given the asymptotic value of 1·2533, and the other constants obtained by fitting the curves by the method of moments to the appropriate six computed points. The equations obtained were as follows:

(*a*) n odd,

$$y = 1{\cdot}2533 - {\cdot}2653/n - {\cdot}0699/n^2 + {\cdot}0822/n^3,$$

where $y = \sigma_m \Big/ \dfrac{\sigma}{\sqrt{n}}$.

(*b*) n even,

$$y = 1{\cdot}2533 - {\cdot}8261/n + {\cdot}7826/n^2 - {\cdot}3478/n^3 + {\cdot}1304/n^4.$$

In this case the equation of third degree was not satisfactory so that a term in n^{-4} was added.

The values of c_m and $c_{m'}$ computed from these curves are given in Tables II[c] and II[d]. The fit is admirable in the case of c_m and probably quite adequate in the case of $c_{m'}$. In the case of n odd K. P.'s formula (given in Appendix) enables us to link

TABLE II[c].

c_m from smooth curve for n = odd.

n	c_m			n	c_m	
	From curve	True value	K. P.'s formula*		From curve	K. P.'s formula*
1	1·0003	1·0000 ($\Delta = +{\cdot}0003$)	—	17	1·2375	1·2372 ($\Delta = {\cdot}0003$)
3	1·1601	1·1602 ($\Delta = -{\cdot}0001$)	1·1550 ($\Delta = +{\cdot}0052$)	19	1·2392	—
5	1·1981	1·1976 ($\Delta = +{\cdot}0005$)	1·1964 ($\Delta = +{\cdot}0012$)	25	1·2426	1·2424 ($\Delta = {\cdot}0002$)
7	1·2142	1·2137 ($\Delta = +{\cdot}0005$)	1·2133 ($\Delta = +{\cdot}0004$)	[31	1·2445]‡	—
9	1·2231	1·2226† ($\Delta = +{\cdot}0005$)	1·2225 ($\Delta = +{\cdot}0001$)	35	1·2457	—
11	1·2287	1·2286 ($\Delta = +{\cdot}0001$)	1·2282 ($\Delta = +{\cdot}0004$)	[41	1·2467]‡	—
13	1·2325	—	—	45	1·2474	—
15	1·2353	—	1·2350 ($\Delta = +{\cdot}0003$)	55	1·2485	1·2484 ($\Delta = {\cdot}0001$)
				[81	1·2499]‡	
				101	1·2506[3]	1·2506[4] ($\Delta = {\cdot}0000$)
				[101	1·2506]‡	
				[161	1·2516]‡	1·2516 ($\Delta = {\cdot}0000$)
				∞	1·2533	1·2533

Table II[c] shows that the K. P. formula gives excellent results from $n = 11$ onward, and might for many purposes suffice from $n = 7$.

up satisfactorily the small to the large samples. For an even n formula (1·2) is not so satisfactory, though it would be adequate for most purposes when the size of the sample was 100 or over.

(3) *The Results of Experimental Sampling.*

It was considered that experimental sampling would be of value both in providing a check on the theoretical work and in bridging over the gap between the distributions in very small samples and the limiting forms. Tippett's Random Numbers

* See Appendix for this formula.

† [I think the true value here ought to be 1·2228, having regard to Professor Hojo's curve and my formula. Ed.]

‡ [The numbers in square brackets are additional true values computed by Mr E. C. Fieller. Each is obtained independently by the quadrature of $\int_0 a_x^p (1-a_x)^p x^q da_x$. Ed.]

TABLE II[a].

$c_{m'}$ from smooth curve for n = even.

n	$c_{m'}$		Δ	n	$c_{m'}$ from curve	Value of limiting formula (1·2)
	From curve	True value				
2	1·0006	1·0000	+·0006	30	1·2266	1·2333 (Δ = −·0067)
4	1·0908	1·0922	−·0014	40	1·2331	1·2382 (Δ = ·0051)
6	1·1358	1·1351	+·0007	50	1·2371	1·2411 (Δ = −·0040)
8	1·1616	1·1600	+·0016	60	1·2398	—
10	1·1782	1·1768	+·0014	70	1·2417	—
12	1·1897	1·1898	−·0001	80	1·2431	—
14	1·1982	—	—	90	1·2442	—
16	1·2047	—	—	100	1·2451	1·2471 (Δ = −·0020)
18	1·2098	—	—			
20	1·2139	—	—	∞	1·2533	1·2533

were therefore used to obtain 500 samples of 30 and 40 from a normal population*. Dr E. S. Pearson also had available 1000 samples of 3, 4, 7, 10, 15 and 20 from a normal population, and from these I extracted the median and quartiles. The results for the median (m and m') are summarised below, and compared with theoretical values.

TABLE III[a].

For m' (n even).

	Size of Sample, n / Number of Samples, N	4 / 1000	10 / 1000	20 / 1000	30 / 500	40 / 500
(1)	$\sigma_{m'} \Big/ \frac{\bar{\sigma}}{\sqrt{n}} = c_{m'}$ Experiment	1·1379	1·1495	1·1707	1·2317	1·2138
(2)	$\sigma_{m'} \Big/ \frac{\bar{\sigma}}{\sqrt{n}} = c_{m'}$ Theory	1·0922	1·1768	1·2139	1·2266	1·2331
(3)	$\{c_{m'}(E) - c_{m'}(T)\}$/S.E.	+1·871	−1·038	−1·592	+0·131	−0·495
(4)	$_{m'}\beta_2$ Experiment	3·2551	2·7810	2·9097	3·0558	2·8754
(5)	$_{m'}\beta_2$ Theory	3·0191	3·0000†	3·0000†	3·0000†	3·0000†
(6)	$\{_{m'}\beta_2(E) - _{m'}\beta_2(T)\}$/S.E.	+1·534	−1·424	−0·587	+0·259	−0·577

* The grouping unit of the population sampled was $\frac{1}{10}$th the standard deviation.

† Theoretical values of 3 have here been given to β_2, though from Table II[a] it is seen that this limit will not be reached by n=10.

TABLE III[b].

For m (n odd).

	Size of Sample, n Number of Samples, N	3 1000	7 1000	15 1000
(1)	$\sigma_m \Big/ \frac{\sigma}{\sqrt{n}} = c_m$ Experiment	1·1321	1·2356	1·1941
(2)	$\sigma_m \Big/ \frac{\sigma}{\sqrt{n}} = c_m$ Theory	1·1602	1·2137	1·2353
(3)	$\{c_m(E) - c_m(T)\}$/S.E.	−1·083	+0·807	−1·492
(4)	${}_m\beta_2$ Experiment	3·0478	2·6726	2·9797
(5)	${}_m\beta_2$ Theory	3·0347	3·0272	3·0000*
(6)	$\{{}_m\beta_2(E) - {}_m\beta_2(T)\}$/S.E.	+0·852	−2·306	−0·132

In these tables row (1) contains the ratio of the experimental value of median to the standard error of the mean or $c_m(E)$, and row (2) the theoretical values of this ratio $c_m(T)$ from Table II[b] or for $n > 12$ from the curves of pp. 327—328. Rows (4) and (5) contain the experimental and theoretical values of β_2. To judge the significance in the differences between experiment and theory, these differences have been divided by their appropriate standard errors in rows (3) and (6). As the distribution of the median is approximately normal the standard error of c_m has been taken as $c_m(T)/\sqrt{2N}$, where N is the number of samples (i.e. 1000 or 500). The standard error used for β_2 was also that appropriate for samples from a normal population; for a sample of 1000 this is ·1538 and of 500, ·2158†. It will be seen that except in the case of ${}_m\beta_2$ when $n = 7$, the experimental values differ by less than twice the standard error from the theoretical. Nor does there appear to be any systematic divergence, positive and negative differences occurring in an apparently random fashion. As would be expected positive deviations in σ_m are associated with positive deviations in ${}_m\beta_2$.

In three cases normal curves were fitted to the experimental distribution of the median and the goodness of fit tested, with the following results:

$$n = 10,\ P_{\chi^2} = \cdot 890;\quad n = 15,\ P_{\chi^2} = \cdot 924;\quad n = 20,\ P_{\chi^2} = \cdot 953.$$

The experimental values of c_m and $c_{m'}$, as well as the theoretical values and the fitted curves, are shown in Fig. 1.

This figure indicates that as far as probably reasonable samples are concerned great accuracy in the values of c_m and $c_{m'}$ are unnecessary, and that after $n = 50$ it may well be doubted whether for practical purposes it is worth while to have separate formulae for n even and n odd.

A diagrammatic representation of the manner in which the curves of β_2 may be supposed to run when n is even and when n is odd is given in Fig. 2.

* Theoretical values of 3 have here been given to β_2, though from Table II[a] it is seen that this limit will not be reached by $n = 10$.

† Values taken from *Biometrika*, Vol. XXII. p. 244.

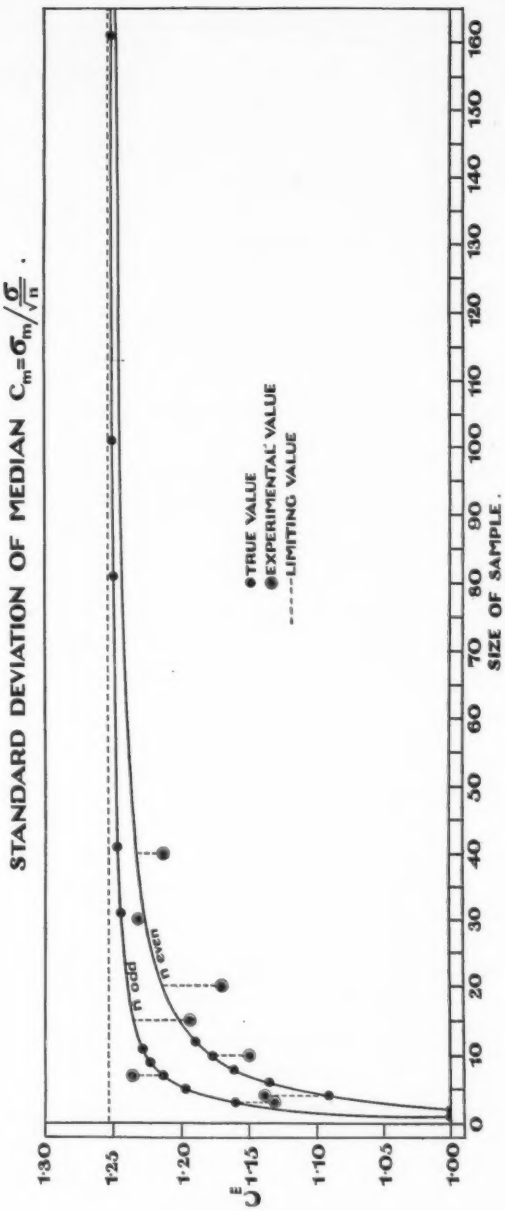
STANDARD DEVIATION OF MEDIAN $C_m=\sigma_m/\frac{\sigma}{\sqrt{n}}$.
1·30
1·25
1·20
1·15
1·10
1·05
1·00
E
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
SIZE OF SAMPLE.
n odd
n even
●TRUE VALUE
◉EXPERIMENTAL VALUE
-----LIMITING VALUE

FIG. 1.

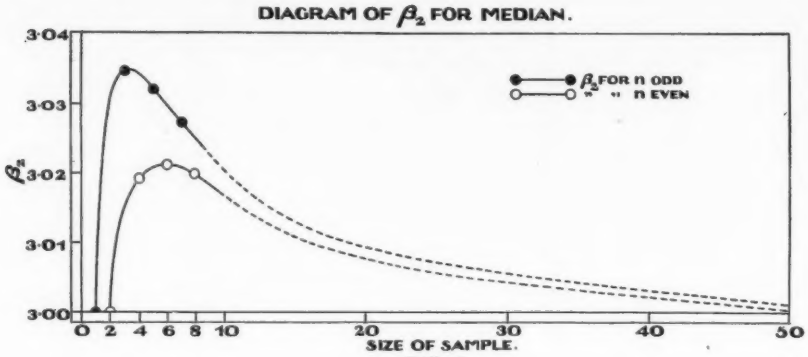

FIG. 2.

II. *The Quartiles.*

It is first necessary to define the sample estimate of the position of the quartiles in the population. Four cases arise according as

(i) $n=4m$, (ii) $n=4m+1$, (iii) $n=4m+2$, (iv) $n=4m+3$.

The position is made clear in the following diagram where the simple case with $m=2$ is illustrated:

Case (i). $n=8\ (=4m)$.

1st quartile Median 3rd quartile

1st quartile at $\frac{1}{2}(x_m+x_{m+1})$; Median at $\frac{1}{2}(x_{2m}+x_{2m+1})$.

Case (ii). $n=9\ (=4m+1)$.

1st quartile Median 3rd quartile

1st quartile at $\frac{1}{2}(x_m+x_{m+1})$; Median at x_{2m+1}.

Case (iii). $n=10\ (=4m+2)$.

1st quartile Median 3rd quartile

1st quartile at x_{m+1}; Median at $\frac{1}{2}(x_{2m+1}+x_{2m+2})$.

Case (iv). $n=11\ (=4m+3)$.

1st quartile Median 3rd quartile

1st quartile at x_{m+1}; Median at x_{2m+2}.

(1) *Limiting Values of the Moment Coefficients of the Quartiles.*

Cases (iii) $n=4m+2$ and (iv) $n=4m+3$.

Here the quartile falls on an observation. The appropriate expression for the standard error, σ_q, when $n\to\infty$, has been given by K. Pearson*, namely

$$\sigma_q=\tilde{\sigma}\sqrt{\tfrac{1}{4}\cdot\tfrac{3}{4}}/z_{\tilde{q}}\sqrt{n}=1{\cdot}3626\tilde{\sigma}/\sqrt{n} \quad \text{.....(15),}$$

* *Loc. cit.* pp. 117 and 118.

where $z_{\bar{q}}$ is the ordinate of the normal curve with unit standard deviation at the population quartile, or $\bar{q} = \cdot 67449$. Following the method employed on p. 316 above we can obtain the limiting forms of ${}_q\beta_1$ and ${}_q\beta_2$. We put in equations (1·1) and (1·2), $s/n = \frac{1}{4}$, $t = q$, and find

$$ {}_q\beta_1 = \frac{\left(1 - \frac{2s}{n}\right)^2}{n\frac{s}{n}\left(1 - \frac{s}{n}\right)} = \frac{4}{3n} \to 0 \quad \text{.....(16),} $$

$$ {}_q\beta_2 = 3\frac{n-2}{n} + \frac{1}{n\frac{s}{n}\left(1 - \frac{s}{n}\right)} = 3 - \frac{2}{3n} \to 3 \quad \text{.....(17),} $$

so that the distribution of q tends to be normal as n increases.

Cases (i) $n = 4m$ and (ii) $n = 4m + 1$.

Here the quartile is at the mid-point between two observations, which for simplicity we shall write as x_1 and x_2,

$$ q' = \tfrac{1}{2}(x_1 + x_2) \qquad \sigma_{q'}^2 = \tfrac{1}{4}(\sigma_{x_1}^2 + \sigma_{x_2}^2 + 2r_{x_1x_2}\sigma_{x_1}\sigma_{x_2}). $$

The required expression for the standard error, when $n \to \infty$, can be obtained from equation (20) of K. Pearson's paper, which in present notation takes the form

$$ \sigma_{q'}^2 = \tfrac{1}{4}\frac{\tilde{\sigma}^2}{n}\left\{\frac{1}{H_1^2}\frac{s_1}{n}\left(1 - \frac{s_1}{n}\right) + \frac{1}{H_2^2}\frac{s_2}{n}\left(1 - \frac{s_2}{n}\right) + \frac{2}{H_1H_2}\frac{s_2}{n}\left(1 - \frac{s_1}{n}\right)\right\} \text{ ...(18),} $$

where

$$ s_1/n = \tfrac{3}{4} + \tfrac{1}{2}n, \qquad s_2/n = \tfrac{3}{4} - \tfrac{1}{2}n $$

$$ H_1^2 = z_1^2(1 - \tfrac{1}{2}n), \qquad H_2^2 = z_2^2(1 + \tfrac{1}{2}n) $$

and z_1 and z_2 are obtained from entering Sheppard's Tables with

$$ \tfrac{1}{2}(1 + \alpha_1) = \tfrac{3}{4} + \tfrac{1}{2}n, \qquad \tfrac{1}{2}(1 + \alpha_2) = \tfrac{3}{4} - \tfrac{1}{2}n. $$

On substituting into (18) it is found that

$$ \sigma_{q'} = \frac{\tilde{\sigma}}{2\sqrt{n}}\sqrt{\frac{3}{16}}\left\{\frac{1}{z_{q_1}^2}\left(1 - \frac{5}{6n} - \frac{7}{4n^2}\right) + \frac{1}{z_{q_2}^2}\left(1 + \frac{5}{6n} - \frac{7}{4n^2}\right) + \frac{2}{z_{q_1}z_{q_2}}\left(1 - \frac{8}{3n} + \frac{35}{24n^2}\right)\right\}^{\frac{1}{2}} $$

.....(18·1).

If $n = 50$, $\sigma_{q'} = 1{\cdot}3453\tilde{\sigma}/\sqrt{n}$; if $n = 100$, $\sigma_{q'} = 1{\cdot}3538\tilde{\sigma}/\sqrt{n}$, the coefficient approaching the value of 1·3626 of Cases (iii) and (iv).

We have not considered the values for β_1 and β_2, but there is little doubt that these will also tend to 0 and 3 as n increases.

(2) *Theoretical Values of the Moment Coefficients of the Quartiles.*

There will be the four cases considered above.

(*a*) Cases (iii) $n = 4m + 2$ and (iv) $n = 4m + 3$.

The mean value in samples of the quartile, $\bar{q}$, is given by the following formula, if

$$ n = 4m + 2, $$

$$ \bar{q} = \frac{n!}{(3m+1)!\,m!}\int_{-\infty}^{+\infty} \alpha_x^{3m+1}(1 - \alpha_x)^m x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx, $$

and if $n = 4m + 3$,

$$\overline{q} = \frac{n!}{(3m+2)!\,m!}\int_{-\infty}^{+\infty} \alpha_x^{3m+2}(1-\alpha_x)^m x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

so that in both cases we may write

$$\overline{q} = \frac{n!}{p!\,m!}\int_{-\infty}^{+\infty} \alpha_x^{p}(1-\alpha_x)^m x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

where $p = 3m+1$ for $n = 4m+2$, $p = 3m+2$ for $n = 4m+3$. On expansion

$$\begin{aligned}\overline{q} &= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \int_{-\infty}^{+\infty} \alpha_x^{p+r} x \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx \\ &= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \left\{\left[-\alpha_x^{p+r}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right]_{-\infty}^{+\infty} + \frac{p+r}{2\pi}\int_{-\infty}^{+\infty}\alpha_x^{p+r-1}e^{-x^2}dx\right\} \\ &= -\frac{n!}{p!\,m!\sqrt{2\pi}}\sum_{r=0}^{m}(-1)^r {}_mC_r (p+r) I_{p+r-1} \quad \text{.....(19).}\end{aligned}$$

The second moment coefficient of the quartile about the population mean is

$$\nu_2' = \frac{n!}{p!\,m!}\int_{-\infty}^{+\infty} \alpha_x^{p}(1-\alpha_x)^m x^2 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

where $p = 3m+1$ for $n = 4m+2$, $p = 3m+2$ for $n = 4m+3$,

$$= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \int_{-\infty}^{+\infty} \alpha_x^{p+r} x^2 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

similarly as in (5),

$$\begin{aligned}&= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \left\{\frac{1}{p+r+1} + \frac{(p+r)(p+r-1)}{4\pi}T_{p+r-2}\right\} \\ &= 1 + \frac{n!}{p!\,m!\,4\pi}\sum_{r=0}^{m}(-1)^r {}_mC_r (p+r)(p+r-1)\,T_{p+r-2} \quad \text{.....(20).}\end{aligned}$$

The third moment coefficient of the quartile about the population mean is

$$\nu_3' = \frac{n!}{p!\,m!}\int_{-\infty}^{+\infty} \alpha_x^{p}(1-\alpha_x)^m x^3 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

where $p = 3m+1$ for $n = 4m+2$, $p = 3m+2$ for $n = 4m+3$,

$$\begin{aligned}&= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \int_{-\infty}^{+\infty} \alpha_x^{p+r} x^3 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx \\ &= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \left\{\left[-\alpha_x^{p+r}x^2\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right]_{-\infty}^{+\infty}\right. \\ &\qquad \left. + \int_{-\infty}^{+\infty}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\left((p+r)\alpha_x^{p+r-1}x^2\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + 2\alpha_x^{p+r}x\right)dx\right\},\end{aligned}$$

which, on reduction in the manner previously indicated, becomes

$$= \frac{n!}{p!\,m!\,2\sqrt{2\pi}}\sum_{r=0}^{m}(-1)^r {}_mC_r (p+r)\left\{5I_{p+r-1} + \frac{(p+r-1)(p+r-2)}{6\pi}K_{p+r-3}\right\}$$

.....(21),

where

$$K_{p+r-3} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_x^{p+r-3}e^{-2x^2}dx.$$

The fourth moment coefficient of the quartile about the population mean is

$$\nu_4' = \frac{n!}{p!\,m!}\int_{-\infty}^{+\infty} \alpha_x^{\,p}(1-\alpha_x)^m x^4 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

where p is as before,

$$= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \int_{-\infty}^{+\infty} \alpha_x^{\,p+r} x^4 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx,$$

similarly as in (6),

$$= \frac{n!}{p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r \left\{\frac{3}{p+r+1} + \frac{13(p+r)(p+r-1)}{12\pi} T_{p+r-2} + \frac{(p+r)(p+r-1)(p+r-2)(p+r-3)}{96\pi^2} S_{p+r-4}\right\},$$

$$\nu_4' = 3 + \frac{n!}{12\pi\,p!\,m!}\sum_{r=0}^{m}(-1)^r {}_mC_r (p+r)(p+r-1) \times \left\{13T_{p+r-2} + \frac{(p+r-2)(p+r-3)}{8\pi} S_{p+r-4}\right\} \quad \text{.....(22).}$$

(*b*) Cases (i) $n = 4m$ and (ii) $n = 4m+1$.

In this case the quartile is defined as

$$q' = \tfrac{1}{2}(x_1 + x_2),$$

where x_1 is mth point and x_2 is $m+1$th point.

The mean of quartile is

$$\bar{q} = \tfrac{1}{2}({}_{x_1}\nu_1' + {}_{x_2}\nu_1') \quad \text{.....(23),}$$

where ${}_{x_1}\nu_1'$ and ${}_{x_2}\nu_2'$ are the means of x_1 and x_2,

$${}_{x_1}\nu_1' = \frac{n!}{p!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1} \alpha_{x_2}^{\,p}(1-\alpha_{x_1})^{m-1} x_1 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2\,dx_1,$$

where $p = 3m-1$ for $n = 4m$ and $p = 3m$ for $n = 4m+1$,

$$= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r {}_{m-1}C_r \int_{-\infty}^{+\infty}\int_{-\infty}^{x_1} \alpha_{x_1}^{\,r}\alpha_{x_2}^{\,p} x_1 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2\,dx_1,$$

and upon reduction

$$= \frac{n!}{(p+1)!\,(m-1)!\sqrt{2\pi}}\sum_{r=0}^{m-1}(-1)^r {}_{m-1}C_r (p+r+1) I_{p+r} \quad \text{.....(24·1),}$$

$${}_{x_2}\nu_1' = \frac{n!}{p!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1} \alpha_{x_2}^{\,p}(1-\alpha_{x_1})^{m-1} x_2 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2\,dx_1$$

$$= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r {}_{m-1}C_r \int_{-\infty}^{+\infty}\int_{-\infty}^{x_1} \alpha_{x_1}^{\,r}\alpha_{x_2}^{\,p} x_2 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2\,dx_1,$$

and upon reduction,

$$= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r \frac{{}_{m-1}C_r}{r+1}\left(\frac{p}{\sqrt{2\pi}} I_{p-1} - \frac{p+r+1}{\sqrt{2\pi}} I_{p+r}\right)$$

$$= \frac{n!}{(p-1)!\,m!\sqrt{2\pi}} I_{p-1} - \frac{n!}{p!\,(m-1)!\sqrt{2\pi}}\sum_{r=0}^{m-1}(-1)^r \frac{{}_{m-1}C_r}{r+1}(p+r+1) I_{p+r} \quad \text{.....(24·2).}$$

On substituting (24·1) and (24·2) in (23), we obtain

$$\overline{q} = \frac{n!}{(p-1)!\,m!\,2\sqrt{2\pi}}\, I_{p-1} - \frac{n!}{(p+1)!\,(m-1)!\,2\sqrt{2\pi}} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \frac{(p+r+1)(p-r)}{r+1}\, I_{p+r} \quad \ldots(24).$$

The second moment coefficient about the population mean is

$$\tfrac{1}{2}(x_1+x_2)\,\nu_2' = \tfrac{1}{4}({}_{x_1}\nu_2' + 2p_{11}' + {}_{x_2}\nu_2') \quad \ldots\ldots\ldots\ldots\ldots\ldots(25),$$

$${}_{x_1}\nu_2' = \frac{n!}{p!\,(m-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_2}^{p} (1-\alpha_{x_1})^{m-1} x_1^{\,2} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_2\, dx_1,$$

where p is as before,

$$= \frac{n!}{p!\,(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_1}^{r} \alpha_{x_2}^{p} x_1^{\,2} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_2\, dx_1,$$

which, as in (5), reduces to

$$= 1 + \frac{n!}{(p+1)!\,(m-1)!\,4\pi} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r (p+r+1)(p+r)\, T_{p+r-1} \quad \ldots\ldots(26{\cdot}1),$$

$$p_{11}' = \frac{n!}{p!\,(m-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_2}^{p} (1-\alpha_{x_1})^{m-1} x_1 x_2 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_2\, dx_1,$$

which, as in (8), reduces to

$$= \frac{n!}{p!\,(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \left\{ \frac{p-r}{4\pi}\, T_{p+r-1} + \frac{pr}{2\pi} (I_{r-1} I_{p-1} - {}_{p-1}I_{r-1}) \right\} \quad \ldots\ldots(26{\cdot}2),$$

$${}_{x_1}\nu_2' = \frac{n!}{p!\,(m-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_2}^{p} (1-\alpha_{x_1})^{m-1} x_2^{\,2} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_2\, dx_1$$

$$= 1 + \frac{n!}{p!\,(m-1)!\,4\pi} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \times \left[\frac{p(p-1)}{r+1}\, T_{p-2} - \left\{ 2p + r + \frac{p(p-1)}{r+1} \right\} T_{p+r-1} \right] \quad \ldots\ldots(26{\cdot}3).$$

Then substituting (26·1), (26·2) and (26·3) in (25), we obtain

$${}_{\frac{1}{2}(x_1+x_2)}\nu_2' = \tfrac{1}{2} + \frac{n!}{(p-2)!\,m!\,16\pi}\, T_{p-2} + \frac{n!}{p!\,(m-1)!\,16\pi} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \times \left[\left\{ \frac{(p+r+1)(p+r)}{p+1} - \frac{p(p-1)}{r+1} - 3r \right\} T_{p+r-1} + 4p r_{r-1} I_{p-1} \right] \quad \ldots(26).$$

The third moment coefficient can be found from the following relation:

$$\tfrac{1}{2}(x_1+x_2)\,\nu_3' = \tfrac{1}{8}({}_{x_1}\nu_3' + 3p_{21}' + 3p_{12}' + {}_{x_2}\nu_3') \quad \ldots\ldots\ldots\ldots\ldots\ldots(27),$$

$${}_{x_1}\nu_3' = \frac{n!}{p!\,(m-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_2}^{p} (1-\alpha_{x_1})^{m-1} x_1^{\,3} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\, dx_2\, dx_1,$$

where $p=3m-1$ for $n=4m$ and $p=3m$ for $n=4m+1$, and accordingly

$$
\begin{aligned}
{}_{x_1}v_3' &= \frac{n!}{(p+1)!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r+1}x_1{}^3\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_1\\
&= \frac{n!}{(p+1)!\,(m-1)!\,2\sqrt{2\pi}}\\
&\qquad\times\left\{5\,(p+r+1)\,I_{p+r}+\frac{(p+r+1)\,(p+r)\,(p+r-1)}{6\pi}\,K_{p+r-2}\right\},\\
{}_{x_2}v_3' &= \frac{n!}{p!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_2}^{p}(1-\alpha_{x_1})^{m-1}x_2{}^3\,\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2dx_1\\
&= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{r}\alpha_{x_2}^{p}x_2{}^3\,\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2dx_1\\
&= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\left\{\left[\frac{1}{r+1}\,\alpha_{x_1}^{r+1}\int_{-\infty}^{x_1}\alpha_{x_2}^{p}x_2{}^3\,\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\,dx_2\right]_{-\infty}^{+\infty}\right.\\
&\qquad\left.-\frac{1}{r+1}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r+1}x_1{}^3\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_1\right\}\\
&= \frac{n!}{p!\,(m-1)!\,2\sqrt{2\pi}}\sum_{r=0}^{m-1}\frac{(-1)^r\,{}_{m-1}C_r}{r+1}\left\{5pI_{p-1}+\frac{p\,(p-1)\,(p-2)}{6\pi}\,K_{p-3}\right.\\
&\qquad\left.-5\,(p+r+1)\,I_{p+r}-\frac{(p+r+1)\,(p+r)\,(p+r-1)}{6\pi}\,K_{p+r-2}\right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
{}_{x_1}v_3'+{}_{x_2}v_3' &= \frac{n!}{(p-1)!\,m!\,2\sqrt{2\pi}}\left(5I_{p-1}+\frac{(p-1)(p-2)}{6\pi}\,K_{p-3}\right)\\
&\qquad-\frac{n!}{(p+1)!\,(m-1)!\,2\sqrt{2\pi}}\sum_{r=0}^{m-1}\frac{(-1)^r\,{}_{m-1}C_r}{r+1}\\
&\qquad\times\left\{5\,(p-r)\,(p+r+1)\,I_{p+r}+\frac{(p^2-r^2)((p+r)^2-1)}{6\pi}\,K_{p+r-2}\right\}\quad\ldots(28{\cdot}1),\\
p_{21}' &= \frac{n!}{p!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_2}^{p}(1-\alpha_{x_1})^{m-1}x_1{}^2x_2\,\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2dx_1\\
&= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{r}\alpha_{x_2}^{p}x_1{}^2x_2\,\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2dx_1,
\end{aligned}
$$

which after some algebraic work reduces to

$$
\begin{aligned}
&= \frac{n!}{p!\,(m-1)!\,\sqrt{2\pi}}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\left\{\frac{p}{(r+1)}\,I_{p-1}-\frac{2p+r+1}{2\,(r+1)}\,I_{p+r}\right.\\
&\qquad\left.+\frac{p\,(2p+5r-2)-(p+r)\,(p+r-1)}{12\pi}\,K_{p+r-2}+\frac{pr\,(r-1)}{4\pi}\,{}_{r-2}H_{p-1}\right\},
\end{aligned}
$$

where $${}_{r-2}H_{p-1}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_{x_1}^{r-2}e^{-\frac{1}{2}x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{p-1}e^{-x_2^2}dx_2,$$

$$
p_{12}' = \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{r}\alpha_{x_2}^{p}x_1x_2{}^2\,\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2dx_1,
$$

and this reduces to

$$p_{12}' = \frac{n!}{p!(m-1)!\,2\sqrt{2\pi}} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \left\{ \frac{p+2r+1}{p+1} I_{p+r} - \frac{(2p+r)(p+r-1)-3p(p-1)}{6\pi} K_{p+r-2} + \frac{p(p-1)r}{2\pi} (T_{p-2} I_{r-1} - {}_{p-2}H_{r-1}) \right\},$$

$$p_{21}' + p_{12}' = \frac{n!}{(p-1)!\,m!\sqrt{2\pi}} I_{p-1} + \frac{n!}{p!(m-1)!\sqrt{2\pi}} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \times \left[\frac{(p-r)(p+r-1)}{6\pi} K_{p+r-2} - \frac{(p-r)(p+r+1)}{(p+1)(r+1)} I_{p+r} + \frac{pr}{4\pi} \{(p-1)(T_{p-2} I_{r-1} - {}_{p-2}H_{r-1}) + (r-1)\,{}_{r-2}H_{p-1}\} \right] \quad \ldots\ldots(28{\cdot}2).$$

On substituting (28·1) and (28·2) in (27), we can find the third moment.

The fourth moment coefficient is

$$\mu_{4(x_1+x_2)} = \tfrac{1}{16} (\mu_{x_1}' + 4p_{31}' + 6p_{22}' + 4p_{13}' + \mu_{x_2}') \quad \ldots\ldots\ldots(29),$$

$$\mu_{x_1}' = \frac{n!}{p!(m-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_2}^{p} (1-\alpha_{x_1})^{m-1} x_1^4 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} dx_2 dx_1,$$

where $p = 3m-1$ for $n = 4m$ and $p = 3m$ for $n = 4m+1$,

$$= \frac{n!}{p!(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_1}^{r} \alpha_{x_2}^{p} x_1^4 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} dx_2 dx_1$$

$$= \frac{n!}{p!(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \frac{1}{p+1} \int_{-\infty}^{+\infty} \alpha_{x_1}^{p+r+1} x_1^4 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} dx_1,$$

$$\mu_{x_2}' = \frac{n!}{p!(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_1}^{r} \alpha_{x_2}^{p} x_2^4 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} dx_2 dx_1$$

$$= \frac{n!}{p!(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \frac{1}{r+1} \left\{ \left[\alpha_{x_1}^{r+1} \int_{-\infty}^{x_1} \alpha_{x_2}^{p} x_2^4 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} dx_2 \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \alpha_{x_1}^{p+r+1} x_1^4 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} dx_1 \right\},$$

$$\mu_{x_1}' + \mu_{x_2}' = \frac{n!}{p!(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \left\{ \frac{1}{r+1} \int_{-\infty}^{+\infty} \alpha_x^{p} x^4 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx + \left(\frac{1}{p+1} - \frac{1}{r+1} \right) \int_{-\infty}^{+\infty} \alpha_x^{p+r+1} x^4 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \right\}$$

$$= \frac{n!}{p!\,m!} \left\{ \frac{3}{p+1} + \frac{13p(p-1)}{12\pi} T_{p-2} + \frac{p(p-1)(p-2)(p-3)}{96\pi^2} S_{p-4} \right\} + \frac{n!}{p!(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \left(\frac{1}{p+1} - \frac{1}{r+1} \right) \times \left\{ \frac{3}{p+r+2} + \frac{13(p+r+1)(p+r)}{12\pi} T_{p+r-1} + \frac{(p+r+1)(p+r)(p+r-1)(p+r-2)}{96\pi^2} S_{p+r-3} \right\},$$

$$\text{or,}\quad {}_{x_1}\nu_4' + {}_{x_2}\nu_4' = 6 + \frac{13n!}{(p-2)!\,m!\,12\pi}T_{p-2} + \frac{n!}{96\pi^2(p-4)!\,m!}S_{p-4}$$
$$- \frac{n!}{(p+1)!\,(m-1)!\,12\pi}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\,\frac{(p^2-r^2)(p+r+1)}{r+1}$$
$$\times\left\{13T_{p+r-1} + \frac{(p+r-1)(p+r-2)}{8\pi}S_{p+r-3}\right\}\cdots\cdots(30{\cdot}1).$$

$$p_{31}' + p_{13}' = \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{r}\alpha_{x_2}^{p}(x_1^3x_2 + x_1x_2^3)$$
$$\times\frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_2\,dx_1,$$

and this reduces to

$$= \frac{n!}{p!\,(m-1)!}\sum_{r=0}^{m-1}{}_{m-1}C_r\left[-\frac{1}{\pi}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r}x_1^3e^{-x_1^2}dx_1\right.$$
$$+\frac{p}{2\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-1}e^{-x_1'^2}dx_1'\int_0^{\infty}\alpha_{x_1'+x_2'}^{r}(x_1'+x_2')^3e^{-\frac{1}{2}(x_1'+x_2')^2}dx_2'$$
$$-\frac{p}{4\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r-1}x_1^2e^{-\frac{1}{2}x_1^2}dx_1 + \frac{p(p-1)}{8\pi^2}$$
$$\times\int_{-\infty}^{+\infty}\left\{\left[-\alpha_{x_2}^{p-2}\frac{e^{-\frac{1}{2}x_2^2}}{3}\right]_{-\infty}^{x_1} + \frac{p-2}{3\sqrt{2\pi}}\int_{-\infty}^{x_1}\alpha_{x_2}^{p-3}e^{-2x_2^2}dx_2\right\}\alpha_{x_1}^{r}x_1e^{-\frac{1}{2}x_1^2}dx_1$$
$$+\frac{p}{4\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1}^{r}\alpha_{x_1-x_2'}^{p-1}x_1e^{-\frac{1}{2}\{x_1^2+2(x_1-x_2')^2\}}dx_2'\,dx_1$$
$$\left.-\frac{1}{\pi}\int_{-\infty}^{+\infty}\alpha_{x_1}^{p+r}x_1e^{-x_1^2} + \frac{p}{\pi\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1}^{r}x_1e^{-\frac{1}{2}x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{p-1}e^{-x_2^2}dx_2\right],$$

in which

$$\int_{-\infty}^{+\infty}\alpha_{x_1}^{r}x_1e^{-\frac{1}{2}x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{p-3}e^{-2x_2^2}dx_2$$
$$= \int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-3}e^{-2x_1'^2}dx_1'\int_0^{\infty}\alpha_{x_1'+x_2'}^{r}(x_1'+x_2')\,e^{-\frac{1}{2}(x_1'+x_2')^2}dx_2'$$
$$= \int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-3}e^{-2x_1'^2}dx_1'\left\{\left[-\alpha_{x_1'+x_2'}^{r}e^{-\frac{1}{2}(x_1'+x_2')^2}\right]_0^{\infty} + \frac{r}{\sqrt{2\pi}}\int_0^{\infty}\alpha_{x_1'+x_2'}^{r-1}e^{-(x_1'+x_2')^2}dx_2'\right\}$$
$$= \sqrt{2\pi}\,\{S_{p+r-3} + r\,(K_{p-3}I_{r-1} - {}_{p-3}K_{r-1})\},$$

$$\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1}^{r}\alpha_{x_1-x_2'}^{p-1}x_1e^{-\frac{1}{2}\{x_1^2+2(x_1-x_2')^2\}}dx_2'\,dx_1$$
$$= \int_{-\infty}^{+\infty}\alpha_{x_1'}^{p-1}e^{-x_1'^2}dx_1'\left\{\left[-\alpha_{x_1'+x_2'}^{r}e^{-\frac{1}{2}(x_1'+x_2')^2}\right]_0^{\infty} + \frac{r}{\sqrt{2\pi}}\int_0^{\infty}\alpha_{x_1'+x_2'}^{r-1}e^{-(x_1'+x_2')^2}dx_2'\right\}$$
$$= \sqrt{2\pi}\,(T_{p+r-1} + r\,{}_{r-1}I_{p-1}),$$

and

$$\int_{-\infty}^{+\infty}\alpha_x^{p+r}xe^{-x^2}dx = \left[-\alpha_x^{p+r}\frac{e^{-x^2}}{2}\right]_{-\infty}^{+\infty} + \frac{p+r}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_x^{p+r-1}e^{-\frac{1}{2}x^2}dx = \frac{p+r}{2}T_{p+r-1},$$

therefore,

$$p_{31}' + p_{13}' = \frac{n!}{p!\,(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r$$

$$\times \left[-\frac{1}{\pi} \left\{ \tfrac{2}{3}(p+r)\, T_{p+r-1} + \frac{(p+r)(p+r-1)(p+r-2)}{48\pi} S_{p+r-3} \right\} \right.$$

$$+ \frac{p}{2\pi} \left\{ \tfrac{7}{3} T_{p+r-1} + \frac{1}{4\pi} \left(\frac{(p+r-1)(p+r-2)}{6} + \frac{r(r-1)}{3} \right.\right.$$

$$\left.\left. + \frac{r(p+r-2)}{4} \right) S_{p+r-3} + \frac{5r}{2} {}_{r-1}I_{p-1} + \frac{r(r-1)(r-2)}{12\pi} {}_{r-3}K_{p-1} \right\}$$

$$- \frac{p}{4\pi} \left\{ \tfrac{1}{3} T_{p+r-1} + \frac{(p+r-1)(p+r-2)}{24\pi} S_{p+r-3} \right\}$$

$$- \frac{p(p-1)}{24\pi^2} \frac{p+r-2}{4} S_{p+r-3} + \frac{p(p-1)(p-2)}{24\pi^2}$$

$$\times \{ S_{p+r-3} + r(K_{p-3} I_{r-1} - {}_{p-3}I_{r-1}) \}$$

$$\left. + \frac{5p}{4\pi} (T_{p+r-1} + r\, {}_{r-1}I_{p-1}) - \frac{p+r}{2\pi} T_{p+r-1} \right],$$

$$= \frac{n!}{2\pi p!\,(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r$$

$$\times \left\{ \frac{7(p-r)}{3} T_{p+r-1} + \frac{p(p-1)(p-r-2) + r(r-1)(p-r+2)}{24\pi} S_{p+r-3} \right.$$

$$+ 5pr\, {}_{r-1}I_{p-1} + \frac{pr(r-1)(r-2)}{12\pi} {}_{r-3}K_{p-1}$$

$$\left. + \frac{p(p-1)(p-2)\,r}{12\pi} (K_{p-3} I_{r-1} - {}_{p-3}K_{r-1}) \right\} \quad \ldots\ldots(30{\cdot}2).$$

Similarly

$$p_{22}' = \frac{n!}{p!\,(m-1)!} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r \int_{-\infty}^{+\infty} \alpha_{x_1}^r \alpha_{x_2}^p x_1^2 x_2^2 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} dx_2 dx_1$$

$$= 1 + \frac{n!}{m!\,(p-2)!\,4\pi} T_{p-2} + \frac{n!}{p!\,(m-1)!\,4\pi} \sum_{r=0}^{m-1} (-1)^r {}_{m-1}C_r$$

$$\times \left[\left\{ \frac{(p+r+1)(p+r)}{p+1} - \frac{5p+4r}{3} - \frac{p(p-1)}{r+1} \right\} T_{p+r-1} \right.$$

$$+ \frac{1}{24\pi} \{ p(p-1)(p+4r-2) - r(r-1)(4p+r-2) \} S_{p+r-3}$$

$$\left. + \frac{p(p-1)\,r(r-1)}{4\pi} {}_{r-2}T_{p-2} \right] \quad \ldots\ldots(30{\cdot}3).$$

On substituting (30·1), (30·2) and (30·3) in (29), we can obtain the fourth moment coefficient.

The integral forms given in Table I above make it possible to calculate from

the preceding general equations the following numerical values for the frequency constants of the sampling distribution of the quartiles:

$$\begin{cases}\text{the mean and standard deviation when } n = 1, 2, \ldots 13, \\ \text{the coefficients } \beta_1 \text{ and } \beta_2 \text{ when } n = 1, 2, \ldots 10.\end{cases}$$

These values, with those of the moment coefficients about the mean, μ_2, μ_3 and μ_4, are given in the following table and Table IV[c]:

TABLE IV[a].

Theoretical Values for Quartiles (unit = population standard deviation).

n	Mean	σ_q	μ_2	μ_3	μ_4
1	·000 000	1·000 00	1·000 0000	·000 000	3·000 000
2	·564 190	·825 65	·681 6901	·077 079	1·422 797
3	·846 284	·747 98	·559 4672	·089 199	·975 522
4	·663 194	·579 52	·335 8387	·015 517	·341 260
5	·828 991	·549 48	·301 9300	·017 685	·276 706
6	·641 755	·528 75	·279 5775	·020 400	·240 238
7	·757 375	·506 69	·256 7307	·021 057	·203 505
8	·662 521	·437 77	·191 6438	·007 275	·111 510
9	·752 135	·424 36	·180 0774	·007 669	·098 513
10	·656 053	·418 35	·175 0143	·008 713	·093 961
11	·728 847	·407 08	·165 7131	—	—
12	·664 627	·367 77	·135 2561	—	—
13	·726 366	·359 51	·129 2464	—	—

(3) *Comparison of Theoretical and Experimental Sampling Results.*

If we consider separately the four cases (i), (ii), (iii) and (iv) we can see more closely the manner of approach to the limiting forms. The figures are set out in Tables IV and V and in the accompanying diagrams. The same sampling material was used as in the case of the median. The number of samples available in each case was either 1000 or 500, but even in the first event the sampling errors involved are of the same order as the differences between the small sample theoretical values and the limiting values of the constants. Hence, while the experimental results provide a general confirmation of the accuracy of theory, they are not of assistance in determining the exact form of the approach to the limit.

The Mean Quartile.

Fig. 3 shows how in the different cases the four curves approach the limiting value of $\cdot 67449\tilde{\sigma}$. Circles representing experimental results are shown for $n = 4$, 10, 20, 30 and 40. It is clear from the form of the curves that the nearest approach of $\bar{q}$ to $\cdot 67449\tilde{\sigma}$ is obtained for small samples when $n = 4m$. This is a further illustration of the better results obtained when median and quartiles are found from *two* individuals. The experimental values are taken from Table V.

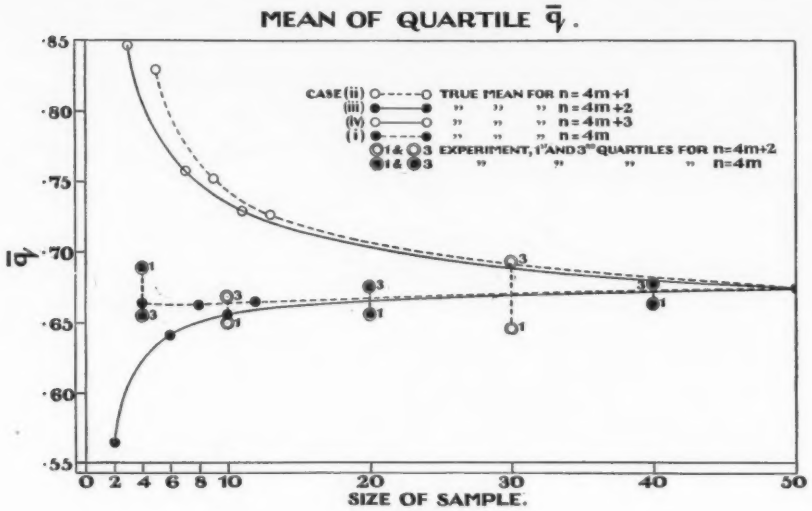
MEAN OF QUARTILE $\bar{q}$.
CASE (ii) TRUE MEAN FOR n = 4m+1
(iii) " " " n = 4m+2
(iv) " " " n = 4m+3
(i) " " " n = 4m
1 & 3 EXPERIMENT, 1ST AND 3RD QUARTILES FOR n=4m+2
1 & 3 " " " " n=4m
$\bar{q}$
·85
·80
·75
·70
·65
·60
·55
0 2 4 6 8 10 20 30 40 50
SIZE OF SAMPLE.

FIG. 3

The Standard Error of the Quartile.

Fig. 4 shows the position for the ratios c_q and $c_{q'}$ of the standard error of the quartile to the standard error of the mean. Here c_q, corresponding to Cases (iii) and (iv), tends more rapidly to the large sample value of 1·3626 than the $c_{q'}$ of

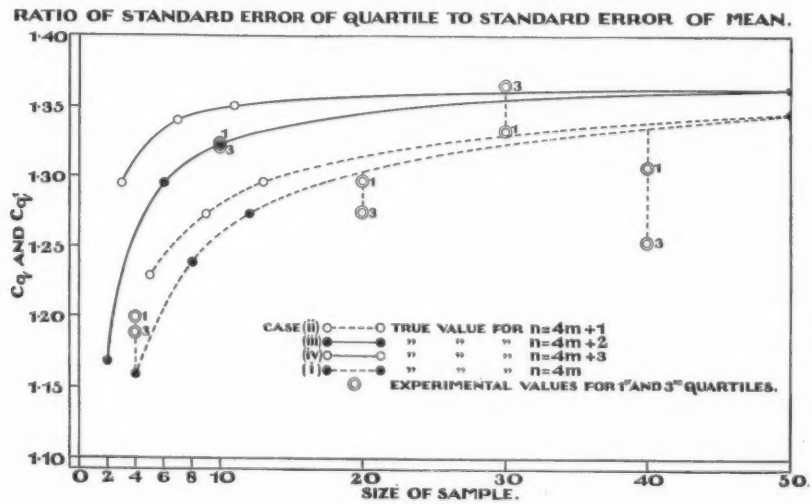

FIG. 4.

Cases (i) and (ii). For $c_{q'}$ the theoretical value at $n = 50$ has been taken from equation (18·1), although there may be some doubt whether this value is accurate enough for a sample of size 50. The third quartile in the case of a sample of 40 gives a $c_{q'}$ of 1·254, while the curve shows 1·335, or there is an error of ·08. The limiting formula for the median's standard error shows at 40, only an error of ·005 on the curve value (see Table II[d]). It seems improbable that the limiting formula for the quartile would give an error sixteen times that for the median, and it is improbable that the curve for the standard error of the quartile could be brought substantially below its present position to be more in harmony with the sample of 40 obtained from experiment.

β_1 and β_2 of Quartiles.

The theoretical values of Table IV[c] are plotted in Fig. 5. For Cases (iii) and

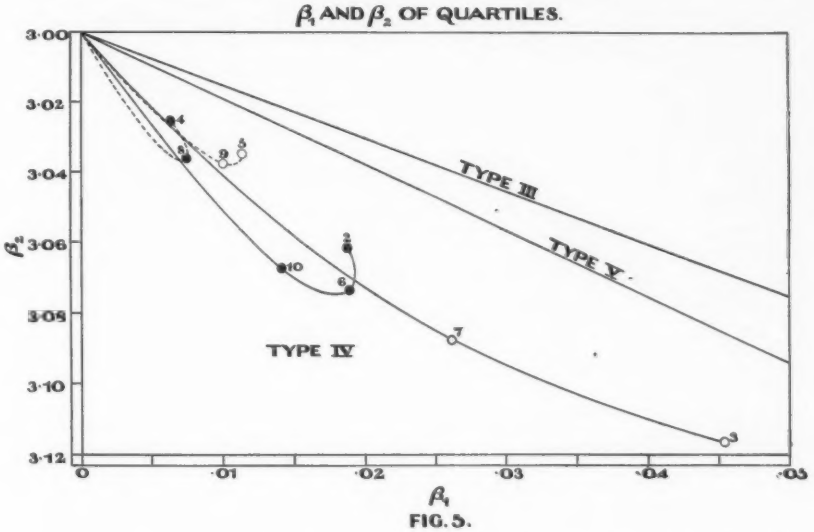

FIG. 5.

(iv) equations (16) and (17) show that $\beta_1 \to 0$ and $\beta_2 \to 3$ as $n \to \infty$. The curves of approach to these limiting values are however purely hypothetical. The experimental values given in Table V are subject to sampling errors which are too large to throw light on the manner of approach to the limit. It seems however likely that for practical purposes we may assume normality in all cases even for small values of n. The results of fitting normal curves to certain of the experimental distributions and applying the P, χ^2 test for goodness of fit are shown in Table V. In all cases examined the fit was satisfactory. The frequency distribution for the combined quartiles was obtained by changing the sign of q_1 and combining with q_2.

TABLE IV[b].

Theoretical Standard Deviation of Quartiles $c_q = \sigma_q \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$ *and* $c_{q'} = \sigma_{q'} \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$.

n	$c_{q'}$	c_q		$c_{q'}$
	Case (ii)	Case (iii)	Case (iv)	Case (i)
1	1·0000			
2		1·1676		
3			1·2955	
4				1·1590
5	1·2287			
6		1·2952		
7			1·3406	
8				1·2382
9	1·2731			
10		1·3229		
11			1·3501	
12				1·2740
13	1·2962			
large	1·3538*	1·3626†	1·3626†	1·3538*

TABLE IV[c].

Theoretical Values for β_1 *and* β_2 *of Quartiles.*

n	Case (ii)		Case (iii)		Case (iv)		Case (i)	
	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
1	·0000	3·0000						
2			·0188	3·0617				
3					·0454	3·1166		
4							·0064	3·0257
5	·0114	3·0353						
6			·0190	3·0735				
7					·0262	3·0876		
8							·0075	3·0362
9	·0101	3·0379						
10			·0142	3·0676				

* This is the value for $c_{q'}$ obtained by taking $n = 100$ in equation (18).

† This is the limiting value of c_q found from equation (15).

TABLE V.

Quartile Distributions. Experimental Values of Frequency Constants.

	Size of Sample, n (No. of Samples, N)	4 (1000)	10 (1000)	20 (1000)	30 (500)	40 (500)
1st quartile q_1	Mean	− ·688 5000	− ·651 2000	− ·656 5000	− ·646 0000	− ·663 2000
	μ_2	·359 0744	·175 3852	·084 1494	·059 2107	·042 7074
	μ_3	− ·041 2483	− ·016 5286	− ·000 5085	·000 8554	− ·002 0287
	μ_4	·397 5065	·100 5950	·020 3024	·011 2949	·005 3798
	β_1	·036 8	·050 6	·000 4	·003 5	·052 8
	β_2	3·083 0	3·270 3	2·867 1	3·221 7	2·949 6
	σ_{q_1}	·599 2	·418 8	·290 1	·243 3	·206 7
	$\sigma_{q_1} \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$	1·198 5	1·324 3	1·297 3	1·332 8	1·307 0
	$P(\chi^2)$ (for Normal)	—	·538	·967	—	—
3rd quartile q_2	Mean	·655 5000	·668 7000	·675 7500	·694 4000	·677 7000
	μ_2	·352 8964	·174 5170	·081 2511	·062 0953	·039 2894
	μ_3	·006 4356	− ·012 1871	·003 2733	·000 8094	− ·000 2943
	μ_4	·376 0634	·094 3956	·019 2245	·012 2666	·004 7272
	β_1	·000 9	·027 9	·020 0	·002 7	·001 4
	β_2	3·019 7	3·099 4	2·912 0	3·181 3	3·062 3
	σ_{q_2}	·594 1	·417 8	·285 0	·249 2	·198 2
	$\sigma_{q_2} \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$	1·188 1	1·321 1	1·274 8	1·364 9	1·253 6
	$P(\chi^2)$ (for Normal)	—	—	·743	—	—
Both quartiles combined	Mean	·672 0000	·659 9500	·666 1250	·670 2000	·670 4500
	μ_2	·356 2577	·175 0277	·082 7929	·061 2386	·041 0510
	μ_3	·023 9948	·014 3464	·001 8491	·000 0817	·000 8301
	μ_4	·388 5153	·097 4997	·019 8626	·012 0748	·005 0327
	β_1	·012 7	·038 4	·006 0	·000 03	·010 0
	β_2	3·061 1	3·182 7	2·897 7	3·219 8	2·986 5
	σ_{q_1,q_2}	·596 9	·418 4	·287 7	·247 5	·202 6
	$\sigma_{q_1,q_2} \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$	1·193 8	1·323 0	2·286 8	1·355 4	1·281 4
	$P(\chi^2)$ (for Normal)	—	—	·858	—	—

III. *The Interquartile Distance.*

This we define as the distance between the two quartile points defined in Section II. There will again be four different cases to consider:

(i) $n = 4m$, (ii) $n = 4m+1$, (iii) $n = 4m+2$, (iv) $n = 4m+3$.

(1) *Limiting Values of the Moment Coefficients of the Quartiles.*

Cases (iii) and (iv). Here K. Pearson has given the standard error of the inter-

quartile distance, $\sigma_{q_1-q_2}$, and also the correlation coefficient between q_1 and q_2, $r_{q_1q_2}$, in the limit for large samples. The results are as follows:

$$\sigma_{q_1-q_2}=\frac{\tilde{\sigma}}{\sqrt{2n}}\frac{1}{z_{\bar{q}}}\sqrt{\tfrac{1}{2}}=2{\cdot}2252\,\frac{\tilde{\sigma}}{\sqrt{2n}}\,^{*} \quad \text{.....(31),}$$

where $z_{\bar{q}}$ is the ordinate at $\bar{q}=\cdot67449$ of a normal curve with unit standard deviation,

$$r_{q_1q_2}=\frac{s}{n}\Big/\left(1-\frac{s}{n}\right)=\tfrac{1}{3}, \quad \text{since } \frac{s}{n}=\tfrac{1}{4} \quad \text{.....(32).}$$

Cases (i) and (ii). Here the quartile estimates are at the mid-point between two observational values, say, $q_1'=\frac{1}{2}(x_1+x_2)$, $q_2'=\frac{1}{2}(x_3+x_4)$.

The mean product of the deviations of q_1' and q_2' from their mean values is

$$\begin{aligned}p_{q_1'q_2'}&=\text{Mean}\left[\delta\left\{\tfrac{1}{2}(x_1+x_2)\right\}\delta\left\{\tfrac{1}{2}(x_3+x_4)\right\}\right]\\&=\tfrac{1}{4}\,\text{Mean}\left[(\delta x_1+\delta x_2)(\delta x_3+\delta x_4)\right]=\tfrac{1}{4}\left\{p_{31}+p_{41}+p_{32}+p_{42}\right\} \quad \text{...(33),}\end{aligned}$$

where

$$\begin{cases}p_{31}=\dfrac{\tilde{\sigma}^2}{nH_3H_1}\dfrac{s_3}{n}\left(1-\dfrac{s_1}{n}\right), & p_{41}=\dfrac{\tilde{\sigma}^2}{nH_4H_1}\dfrac{s_4}{n}\left(1-\dfrac{s_1}{n}\right),\\[2ex] p_{32}=\dfrac{\tilde{\sigma}^2}{nH_3H_2}\dfrac{s_3}{n}\left(1-\dfrac{s_2}{n}\right), & p_{42}=\dfrac{\tilde{\sigma}^2}{nH_4H_2}\dfrac{s_4}{n}\left(1-\dfrac{s_2}{n}\right),\end{cases}$$

and as we are dealing with the two quartiles of a normal population

$$\frac{s_4}{n}=1-\frac{s_1}{n}=\tfrac{1}{4}-\frac{1}{2n}, \quad \frac{s_3}{n}=1-\frac{s_2}{n}=\tfrac{1}{4}+\frac{1}{2n},$$

$$H_1^2=H_4^2=z_1^2\left(1-\frac{1}{2n}\right), \quad H_2^2=H_3^2=z_2^2\left(1+\frac{1}{2n}\right),$$

and z_1 and z_2 are obtained from entering Sheppard's Tables of the normal probability integral with

$$\tfrac{1}{2}(1+a_1)=\tfrac{3}{4}+\tfrac{1}{2}n, \qquad \tfrac{1}{2}(1+a_2)=\tfrac{3}{4}-\tfrac{1}{2}n \text{ respectively.}$$

Using these results in (33) it is found that

$$p_{q_1'q_2'}=\frac{\tilde{\sigma}^2}{64n}\left\{\frac{1}{z_1}\left(1-\frac{7}{4n}-\frac{13}{32n^2}\cdots\right)+\frac{1}{z_2}\left(1+\frac{7}{4n}-\frac{13}{32n^2}\cdots\right)\right\}^2 \quad \text{.....(34).}$$

The standard error of the interquartile distance and the coefficient of correlation between the quartiles are therefore

$$\sigma^2_{q_1'-q_2'}=2\sigma^2_{q'}(1-r_{q_1'q_2'}) \quad \text{.....(35),}$$

where $\sigma_{q'}$ is given by (18) above, and

$$r_{q_1'q_2'}=p_{q_1'q_2'}/\sigma^2_{q'} \quad \text{.....(36).}$$

If $n=50$, these equations give

$$\sigma_{q_1'-q_2'}=2{\cdot}1826\tilde{\sigma}/\sqrt{2n}\,; \quad r_{q_1'q_2'}=\cdot3420.$$

If $n=100$, $\qquad \sigma_{q_1'-q_2'}=2{\cdot}2034\tilde{\sigma}/\sqrt{2n}\,; \quad r_{q_1'q_2'}=\cdot3377.$

These values approach those given in (31) and (32) for Cases (iii) and (iv).

* The interquartile distance is a measure of variation, and hence the factor $\sqrt{2n}$ puts the result in a form comparable with the standard error of the standard deviation for samples from a normal population, or with $\sigma/\sqrt{2n}$.

Values for β_1 and β_2.

The third and fourth moment coefficients of $q_1 - q_2$ will depend upon the third and fourth order product-moment coefficients of q_1 and q_2. Tippett* has considered an analogous problem when dealing with the sampling distribution of the range. He assumed that for large samples the regression of one extreme variate on the other might be considered as linear and the array distributions of the correlation surface as homoscedastic. As shown by E. S. Pearson† this assumption was not justified in the case of small samples, but it appeared to be satisfactory in the case of samples of 50 or more. Following the method of Tippett in the case of q_1 and q_2, we have

$$\left.\begin{aligned} {}_{q_1-q_2}\mu_2 &= 2\,{}_{q_1}\mu_2(1-r) \\ {}_{q_1-q_2}\mu_3 &= 2\,{}_{q_1}\mu_3 + 6p_{12} \\ {}_{q_1-q_2}\mu_4 &= 2\,{}_{q_1}\mu_4 - 8p_{13} + 6p_{22} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(37{\cdot}1),$$

which hold without any assumptions. These formulae will give β_1 and β_2 of $(q_1 - q_2)$ if the moment and product-moment coefficients of q_1 and q_2 up to the fourth order are known. If, further, we make Tippett's assumptions, we obtain

$$p_{12} = -r\,{}_{q_1}\mu_3; \quad p_{13} = r\,{}_{q_1}\mu_4; \quad p_{22} = \{1 + r^2({}_{q_1}\beta_2 - 1)\}\,{}_{q_1}\mu_2^2 \quad \ldots(37{\cdot}2).$$

Whence

$${}_{q_1-q_2}\beta_1 = \tfrac{1}{2}\frac{(1-3r)^2}{(1-r)^3}\,{}_{q_1}\beta_1 \to 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(38{\cdot}1),$$

since $r \to \frac{1}{3}$ and ${}_{q_1}\beta_1 \to 0$ (Equation (16)).

Again

$${}_{q_1-q_2}\beta_2 = \frac{2\,{}_{q_1}\mu_4(1-4r) + 6\{1 + r^2({}_{q_1}\beta_2 - 1)\}\,{}_{q_1}\mu_2^2}{4\,{}_{q_1}\mu_2^2(1-r)^2}$$

$$= \frac{(1-4r)\,{}_{q_1}\beta_2 + 3\{1 + r^2({}_{q_1}\beta_2 - 1)\}}{2(1-r)^2} \to 3 \quad \ldots\ldots\ldots\ldots(38{\cdot}2),$$

since $r \to \frac{1}{3}$ and ${}_{q_1}\beta_2 \to 3$ (Equation (17)).

Hence we should expect in the limit that the sampling distribution of the inter-quartile distance will tend to become normal if the assumptions made above are justified. The work which follows shows that they are not justified at all for small samples. In the case of the range or distance between extreme individuals in the sample, it is interesting to note that Tippett found the limiting form was not normal.

(2) *Theoretical Values of the Moment Coefficients in Small Samples.*

The correlation coefficient $r_{q_1 q_2}$ is

$$r_{q_1 q_2} = (p_{11}' - \bar{q}_1\bar{q}_2)/\sigma_{q_1}\sigma_{q_2} = p_{11}/\sigma_{q_1}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(39),$$

where p_{11}' is the product moment of q_1, q_2, and the standard deviation is

$$\sigma_{q_1-q_2}^2 = 2\sigma_{q_1}^2(1 - r_{q_1 q_2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(40).$$

First, therefore, we must calculate the product moment p_{11}', afterwards we may obtain $r_{q_1 q_2}$ and $\sigma_{q_1-q_2}$ from the above formulae.

* *Biometrika*, Vol. XVII. p. 364.
† *Biometrika*, Vol. XVIII. p. 179.

Cases (iii) $n = 4m+2$ and (iv) $4m+3$.

$$p_{11}' = \frac{n!}{(m!)^2 p!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \alpha_{x_2}^{m} (\alpha_{x_1} - \alpha_{x_2})^p (1-\alpha_{x_1})^m x_1 x_2 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} dx_2 dx_1,$$

where $p = 2m$ for $n = 4m+2$ and $p = 2m+1$ for $n = 4m+3$,

$$= \frac{n!}{(m!)^2 p!} \int_{-\infty}^{+\infty} (1-\alpha_{x_1})^m x_1 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}} \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r \alpha_{x_1}^{p-r} \int_{-\infty}^{+\infty} \alpha_{x_2}^{m+r} x_2 \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} dx_2 \right\} dx_1$$

$$= \frac{n!}{(m!)^2 p!} \int_{-\infty}^{+\infty} (1-\alpha_{x_1})^m x_1 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}$$

$$\times \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r \alpha_{x_1}^{p-r} \left(\left[-\alpha_{x_2}^{m+r} \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \right]_{-\infty}^{x_1} + \frac{m+r}{2\pi} \int_{-\infty}^{x_1} \alpha_{x_2}^{m+r-1} e^{-x_2^2} dx_2 \right) \right\} dx_1$$

$$= \frac{n!}{(m!)^2 p!\, 2\pi} \int_{-\infty}^{+\infty} \sum_{s=0}^{m} (-1)^s {}_mC_s \alpha_{x_1}^{s} x_1 \frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}$$

$$\times \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r \alpha_{x_1}^{p-r} (m+r) \int_{-\infty}^{x_1} \alpha_{x_2}^{m+r-1} e^{-x_2^2} dx_2 \right\} dx_1$$

$$= \frac{n!}{(m!)^2 p!\, 2\pi \sqrt{2\pi}} \sum_{s=0}^{m} (-1)^s {}_mC_s$$

$$\times \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r (m+r) \int_{-\infty}^{+\infty} \alpha_{x_1}^{p+s-r} x_1 e^{-\frac{1}{2}x_1^2} dx_1 \int_{-\infty}^{x_1} \alpha_{x_2}^{m+r-1} e^{-x_2^2} dx_2 \right\},$$

and by changing variables to $x_1' = x_2$, $x_2' = x_1 - x_2$,

$$= \frac{n!}{(m!)^2 p!} \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{s=0}^{m} (-1)^s {}_mC_s \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r (m+r) \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \right.$$

$$\left. \times \int_{0}^{\infty} \alpha_{x_1'+x_2'}^{p+s-r} (x_1' + x_2') e^{-\frac{1}{2}(x_1'+x_2')^2} dx_2' \right\}$$

$$= \frac{n!}{(m!)^2 p!\, (2\pi)^{\frac{1}{2}}} \sum_{s=0}^{m} (-1)^s {}_mC_s \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r (m+r) \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \right.$$

$$\left. \times \left(\left[-\alpha_{x_1'+x_2'}^{p+s-r} e^{-\frac{1}{2}(x_1'+x_2')^2} \right]_0^{\infty} + \frac{(p+s-r)}{\sqrt{2\pi}} \int_0^{\infty} \alpha_{x_1'+x_2'}^{p+s-r-1} e^{-\frac{1}{2}(x_1'+x_2')^2} dx_2' \right) \right\}$$

$$= \frac{n!}{(m!)^2 p!} \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{s=0}^{m} (-1)^s {}_mC_s \left\{ \sum_{r=0}^{p} (-1)^r {}_pC_r m+r \sqrt{2\pi} \right.$$

$$\left. \times [T_{p+m+s-1} + (p+s-r)(I_{m+r-1} I_{p+s-r-1} - {}_{m+r-1}I_{p+s-r-1})] \right\}$$

$$= \frac{n!}{(m!)^2 p!\, 2\pi} \sum_{s=0}^{m} (-1)^s {}_mC_s \sum_{r=0}^{p} (-1)^r {}_pC_r (m+r)(p+s-r) {}_{p+s-r-1}I_{m+r-1}$$

.....(41).

Cases (i) $n = 4m$ and (ii) $4m+1$.

$$p_{11}' = \frac{n!}{(m-1)!^2 p!} \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} \alpha_{x_4}^{m-1} (\alpha_{x_3} - \alpha_{x_2})^p (1-\alpha_{x_1})^{m-1}$$

$$\times \left(\frac{x_1 + x_2}{2} \right) \left(\frac{x_3 + x_4}{2} \right) d\alpha_{x_4} d\alpha_{x_3} d\alpha_{x_2} d\alpha_{x_1},$$

where $p=2m-2$ for $n=4m$ and $p=2m-1$ for $n=4m+1$, and this:

$$=\frac{n!}{4(m-1)!^2p!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}(1-\alpha_{x_1})^{m-1}(\alpha_{x_2}-\alpha_{x_3})^p(x_1+x_2)$$

$$\times\left\{\frac{\alpha_{x_3}^m}{m}x_3-\alpha_{x_3}^{m-1}\frac{e^{-\frac{1}{2}x_3^2}}{\sqrt{2\pi}}+\frac{m-1}{2\pi}\int_{-\infty}^{x_3}\alpha_{x_4}^{m-2}e^{-x_4^2}dx_4\right\}d\alpha_{x_3}d\alpha_{x_2}d\alpha_{x_1}$$

$$=\frac{n!}{4(m-1)!^2p!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}(1-\alpha_{x_1})^{m-1}(x_1+x_2)\sum_{r=0}^{p}(-1)^r{}_pC_r\alpha_{x_2}^{p-r}$$

$$\times\left\{\frac{r}{2\pi m}\int_{-\infty}^{x_2}\overset{[1]}{\alpha}_{x_3}^{m+r-1}e^{-x_3^2}dx_3+\frac{m-1}{(\sqrt{2\pi})^3}\int_{-\infty}^{x_2}\overset{[2]}{\alpha}_{x_3}^{r}e^{-\frac{1}{2}x_3^2}dx_3\right.$$

$$\left.\times\int_{-\infty}^{x_3}\alpha_{x_4}^{m-2}e^{-x_4^2}dx_4\right\}d\alpha_{x_3}d\alpha_{x_2}\ldots\ldots(42\cdot1).$$

First term of (42·1).

$$\frac{1}{4\pi^2m}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}(1-\alpha_{x_1})^{m-1}(x_1+x_2)\sum_{r=0}^{p}(-1)^r{}_pC_r\,r\alpha_{x_2}^{p-r}e^{-\frac{1}{2}x_1^2}e^{-\frac{1}{2}x_2^2}dx_1dx_2$$

$$\times\int_{-\infty}^{x_3}\alpha_{x_3}^{m+r-1}e^{-x_3^2}dx_3$$

$$=\frac{1}{4\pi^2m}\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\sum_{r=0}^{p}(-1)^r{}_pC_r\,r\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_3}\alpha_{x_1}^s\alpha_{x_2}^{p-r}\alpha_{x_3}^{m+r-1}(x_1+x_2)$$

$$\times e^{-\frac{1}{2}(x_1^2+x_2^2+2x_3^2)}dx_3dx_2dx_1,$$

and by changing variables to $x_1'=x_3$, $x_1'+x_3'=x_2$ and $x_1'+x_2'+x_3'=x_1$,

$$=\Phi_{r,s}\int_{-\infty}^{+\infty}\int_0^{\infty}\int_0^{\infty}\alpha_{x_1'}^{m+r-1}\alpha_{x_1'+x_3'}^{p-r}\alpha_{x_1'+x_2'+x_3'}^{s}\{(x_1'+x_2'+x_3')+(x_1'+x_3')\}$$

$$\times e^{-\frac{1}{2}\{(x_1'+x_2'+x_3')^2+(x_1'+x_3')^2+2x_1'^2\}}dx_3'dx_2'dx_1',$$

$\Phi_{r,s}$ standing for the operation

$$\frac{1}{4\pi^2m}\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\sum_{r=0}^{p}(-1)^r{}_pC_r\,r,$$

$$=\Phi_{r,s}\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1'}^{m+r-1}\alpha_{x_1'+x_3'}^{p-r}\left\{\left[-\alpha_{x_1'+x_2'+x_3'}^{s}e^{-\frac{1}{2}(x_1'+x_2'+x_3')^2}\right]_{x_1'=0}^{\infty}\right.$$

$$+\frac{s}{\sqrt{2\pi}}\int_0^{\infty}\alpha_{x_1'+x_2'+x_3'}^{s-1}e^{-(x_1'+x_2'+x_3')^2}dx_2'$$

$$\left.+(x_1'+x_3')\frac{\sqrt{2\pi}}{s+1}\left[\alpha_{x_1'+x_2'+x_3'}^{s+1}\right]_{x_1'=0}^{\infty}\right\}e^{-\frac{1}{2}\{(x_1'+x_3')^2+2x_1'^2\}}dx_3'dx_1',$$

$$=\Phi_{r,s}\left\{\int_{-\infty}^{+\infty}\int_0^{\omega}\alpha_{x_1'}^{m+r-1}\alpha_{x_1'+x_3'}^{p+s-r}e^{-(x_1'+x_3')^2-x_1'^2}dx_3'dx_1'\right.$$

$$+\frac{s}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1'}^{m+r-1}\alpha_{x_1'+x_3'}^{p-r}\left(\sqrt{2\pi}I_{s-1}-\int_{-\infty}^{x_1'+x_3'}\alpha_x^{s-1}e^{-x^2}dx\right)$$

$$\times e^{-\frac{1}{2}\{(x_1'+x_3')^2+2x_1'^2\}}dx_3'dx_1'$$

$$\left.+\frac{\sqrt{2\pi}}{s+1}\int_{-\infty}^{+\infty}\int_0^{\infty}\alpha_{x_1'}^{m+r-1}\alpha_{x_1'+x_3'}^{p-r}\left(1-\alpha_{x_1'+x_3'}^{s+1}\right)(x_1'+x_3')e^{-\frac{1}{2}\{(x_1'+x_3')^2+2x_1'^2\}}dx_3'dx_1'\right\},$$

$$= \Phi_{r,s}\left\{\int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \left(\sqrt{2\pi} I_{p+s-r} - \int_{-\infty}^{x_1'} \alpha_x^{p+s-r} e^{-x^2} dx\right)\right.$$

$$+ \frac{s}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \left(\sqrt{2\pi} I_{s-1} \left[\frac{\sqrt{2\pi}}{p-r+1} \alpha_{x_1'+x_3'}^{p-r+1}\right]_0^\infty \right.$$

$$\left. - \int_0^\infty \alpha_{x_1'+x_3'}^{p-r} e^{-\frac{1}{2}(x_1'+x_3')^2} dx_3' \int_{-\infty}^{x_1'+x_3'} \alpha_x^{s-1} e^{-x^2} dx\right)$$

$$+ \frac{\sqrt{2\pi}}{s+1} \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \left(\left[-\alpha_{x_1'+x_3'}^{p-r} e^{-\frac{1}{2}(x_1'+x_3')^2}\right]_0^\infty \right.$$

$$\left. + \frac{p-r}{\sqrt{2\pi}} \int_0^\infty \alpha_{x_1'+x_3'}^{p-r-1} e^{-(x_1'+x_3')^2} dx_3'\right)$$

$$\left. - \left[-\alpha_{x_1'+x_3'}^{p+s-r+1} e^{-\frac{1}{2}(x_1'+x_3')^2}\right]_0^\infty - \frac{p+s-r+1}{\sqrt{2\pi}} \int_0^\infty \alpha_{x_1'+x_3'}^{p+s-r} e^{-(x_1'+x_3')^2} dx_3'\right)\Bigg\},$$

$$= \Phi_{r,s}\left[2\pi_{p+s-r} I_{m+r-1} + \frac{s}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \left\{\frac{2\pi}{p-r+1} I_{s-1}\left(1 - \alpha_{x_1'}^{p-r+1}\right)\right.\right.$$

$$\left. - \left[\frac{\sqrt{2\pi}\,\alpha_{x_1'+x_3'}^{p-r+1}}{p-r+1} \int_{-\infty}^{x_1'+x_3'} \alpha_x^{s-1} e^{-x^2} dx\right]_{x_3'=0}^\infty + \frac{\sqrt{2\pi}}{p-r+1} \int_0^\infty \alpha_{x_1'+x_3'}^{p+s-r} e^{-(x_1'+x_3')^2} dx_3'\right\}$$

$$+ \frac{\sqrt{2\pi}}{s+1} \left\{\int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} \left(\alpha_{x_1'}^{p-r} - \alpha_{x_1'}^{p+s-r+1}\right) e^{-\frac{3}{2}x_1'^2} dx_1' \right.$$

$$+ \frac{p-r}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} \left(\sqrt{2\pi} I_{p-r-1} - \int_{-\infty}^{x_1'} \alpha_x^{p-r-1} e^{-x^2} dx\right) dx_1'$$

$$\left.\left. - \frac{p+s-r+1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \alpha_{x_1'}^{m+r-1} e^{-x_1'^2} dx_1' \left(\sqrt{2\pi} I_{p+s-r} - \int_{-\infty}^{x_1'} \alpha_x^{p+s-r} e^{-x^2} dx\right)\right\}\right],$$

$$= \Phi_{r,s} 2\pi \left[{}_{p+s-r}I_{m+r-1} + \frac{s}{p-r+1}\{I_{s-1}(I_{m+r-1} - I_{p+m})\right.$$

$$- (I_{s-1} - I_{p+s-r}) I_{m+r-1} + {}_{p+m}I_{s-1} - {}_{m+r-1}I_{p+s-r}\}$$

$$\left. + \frac{1}{s+1}\{T_{p+m-1} - T_{p+m+s} + (p-r)_{p-r-1}I_{m+r-1} - (p+s-r+1)_{p+s-r}I_{m+r-1}\}\right]$$

$$= \frac{1}{2\pi m} \sum_{s=0}^{m-1} (-1)^s {}_{m-1}C_s \left[\sum_{r=0}^{p} (-1)^r {}_pC_r\, r \left\{\left(\frac{s}{p-r+1} - \frac{p-r}{s+1}\right) {}_{p+s-r}I_{m+r-1}\right.\right.$$

$$\left.\left. + \frac{p-r}{s+1} {}_{p-r-1}I_{m+r-1}\right\} - (-1)^p s\, {}_{s-1}I_{p+m}\right] \quad \text{.....(42·2).}$$

Second term of (42·1) is

$$\frac{m-1}{(\sqrt{2\pi})^5} \sum_{s=0}^{m-1} (-1)^s {}_{m-1}C_s \sum_{r=0}^{p} (-1)^r {}_pC_r \int_{-\infty}^{+\infty} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} \alpha_{x_1}^s \alpha_{x_2}^{p-r} \alpha_{x_3}^r \alpha_{x_4}^{m-2} (x_1 + x_2)$$

$$\times e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + 2x_4^2)} dx_4\, dx_3\, dx_2\, dx_1,$$

and by changing variables in a manner which can be clearly followed, and omitting the dashes for convenience,

$$= \Phi'_{r,s} \int_{-\infty}^{+\infty} \int_0^\infty \int_0^\infty \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \alpha_{x_1+x_2+x_3+x_4}^{p-r} \alpha_{x_1+x_2+x_3+x_4}^{s}$$
$$\times \{(x_1+x_2+x_3+x_4)+(x_1+x_3+x_4)\}$$
$$\times e^{-\frac{1}{2}\{(x_1+x_2+x_3+x_4)^2+(x_1+x_2+x_4)^2+(x_1+x_4)^2+2x_1^2\}} dx_4 dx_3 dx_2 dx_1,$$

$\Phi'_{r,s}$ standing for

$$\frac{m-1}{(\sqrt{2\pi})^5} \sum_{s=0}^{m-1} (-1)^s {}_{m-1}C_s \sum_{r=0}^{p} (-1)^r {}_pC_r,$$

$$= \Phi'_{r,s} \int_{-\infty}^{+\infty} \int_0^\infty \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \alpha_{x_1+x_2+x_3+x_4}^{p-r} \left\{ \left[-\alpha_{x_1+x_2+x_3+x_4}^{s} e^{-\frac{1}{2}(x_1+x_2+x_3+x_4)^2} \right]_{x_2=0}^{\infty} \right.$$
$$\left. + \frac{s}{\sqrt{2\pi}} \int_0^\infty \alpha_{x_1+x_2+x_3+x_4}^{s-1} e^{-(x_1+x_2+x_3+x_4)^2} dx_2 + (x_1+x_3+x_4) \frac{\sqrt{2\pi}}{s+1} \left[\alpha_{x_1+x_2+x_3+x_4}^{s+1} \right]_{x_2=0}^{\infty} \right\}$$
$$\times e^{-\frac{1}{2}\{(x_1+x_2+x_3+x_4)^2+(x_1+x_3+x_4)^2+(x_1+x_4)^2+2x_1^2\}} dx_3 dx_4 dx_1,$$

$$= \Phi'_{r,s} \left\{ \int_{-\infty}^{+\infty} \int_0^\infty \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \alpha_{x_1+x_2+x_4}^{p+s-r} e^{-\frac{1}{2}\{2(x_1+x_2+x_4)^2+(x_1+x_4)^2+2x_1^2\}} dx_3 dx_4 dx_1 \right.$$
$$+ \frac{s}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^\infty \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \alpha_{x_1+x_2+x_4}^{p-r} \left(\sqrt{2\pi} I_{s-1} - \int_{-\infty}^{x_1+x_2+x_4} \alpha_x^{s-1} e^{-x^2} dx \right)$$
$$\times e^{-\frac{1}{2}\{(x_1+x_2+x_4)^2+(x_1+x_4)^2+2x_1^2\}} dx_3 dx_4 dx_1$$
$$\left. + \frac{\sqrt{2\pi}}{s+1} \int_{-\infty}^{+\infty} \int_0^\infty \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \alpha_{x_1+x_2+x_4}^{p-r} (x_1+x_3+x_4) \left(1 - \alpha_{x_1+x_2+x_4}^{s+1} \right) \right.$$
$$\left. \times e^{-\frac{1}{2}\{(x_1+x_2+x_4)^2+(x_1+x_4)^2+2x_1^2\}} dx_3 dx_4 dx_1 \right\},$$

$$= \Phi'_{r,s} \left[\int_{-\infty}^{+\infty} \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \left(\sqrt{2\pi} I_{p+s-r} - \int_{-\infty}^{x_1+x_4} \alpha_x^{p+s-r} e^{-x^2} dx \right) \right.$$
$$\times e^{-\frac{1}{2}(x_1+x_4)^2} e^{-x_1^2} dx_4 dx_1$$
$$+ \frac{s}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \left\{ \frac{2\pi}{p-r+1} I_{s-1} \left(1 - \alpha_{x_1+x_4}^{p-r+1} \right) \right.$$
$$- \left[\frac{\sqrt{2\pi}}{p-r+1} \alpha_{x_1+x_2+x_4}^{p-r+1} \int_{-\infty}^{x_1+x_2+x_4} \alpha_x^{s-1} e^{-x^2} dx \right]_{x_2=0}^{\infty}$$
$$\left. + \frac{\sqrt{2\pi}}{p-r+1} \int_0^\infty \alpha_{x_1+x_2+x_4}^{p+s-r} e^{-(x_1+x_2+x_4)^2} dx_2 \right\} e^{-\frac{1}{2}(x_1+x_4)^2} e^{-x_1^2} dx_4 dx_1$$
$$+ \frac{\sqrt{2\pi}}{s+1} \int_{-\infty}^{+\infty} \int_0^\infty \alpha_{x_1}^{m-2} \alpha_{x_1+x_4}^{r} \left\{ \left(\alpha_{x_1+x_4}^{p-r} - \alpha_{x_1+x_4}^{p+s-r+1} \right) e^{-\frac{1}{2}(x_1+x_4)^2} \right.$$
$$+ \frac{p-r}{\sqrt{2\pi}} \left(\sqrt{2\pi} I_{p-r-1} - \int_{-\infty}^{x_1+x_4} \alpha_x^{p-r-1} e^{-x^2} dx \right)$$
$$\left. \left. - \frac{p+s-r+1}{\sqrt{2\pi}} \left(\sqrt{2\pi} I_{p+s-r} - \int_{-\infty}^{x_1+x_4} \alpha_x^{p+s-r} e^{-x^2} dx \right) \right\} e^{-\frac{1}{2}(x_1+x_4)^2} e^{-x_1^2} dx_4 dx_1 \right],$$

$$= \Phi'_{r,s} \left[\int_{-\infty}^{+\infty} \alpha_x^{m-2} e^{-x_1^2} dx_1 \left\{ \frac{2\pi}{r+1} I_{p+s-r} \left[\alpha_{x_1+x_4}^{r+1} \right]_{x_4=0}^{\infty} \right. \right.$$
$$\left. - \left[\frac{\sqrt{2\pi}}{r+1} \alpha_{x_1+x_4}^{r+1} \int_{-\infty}^{x_1+x_4} \alpha_x^{p+s-r} e^{-x^2} dx \right]_{x_4=0}^{\infty} + \frac{\sqrt{2\pi}}{r+1} \int_0^\infty \alpha_{x_1+x_4}^{p+s+1} e^{-(x_1+x_4)^2} dx_4 \right\}$$

+ (continued p. 352)

$$+\frac{s}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1}^{m-2}e^{-x_1^2}dx_1\left\{\frac{2\pi\sqrt{2\pi}}{p-r+1}I_{s-1}\left[\frac{\alpha_{x_1+x_4}^{r+1}}{r+1}-\frac{\alpha_{x_1+x_4}^{p+2}}{p+2}\right]_{x_4=0}^{\infty}\right.$$

$$-\frac{\sqrt{2\pi}}{p-r+1}\left(\frac{2\pi}{r+1}I_{s-1}\left[\alpha_{x_1+x_4}^{r+1}\right]_{x_4=0}^{\infty}-\int_0^{\infty}\alpha_{x_1+x_4}^{p+1}e^{-\frac{1}{2}(x_1+x_4)^2}dx_4\int_{-\infty}^{x_1+x_4}\alpha_x^{s-1}e^{-x^2}dx\right.$$

$$\left.\left.-\int_0^{\infty}\alpha_{x_1+x_4}^{r}\left(\sqrt{2\pi}\,I_{p+s-r}-\int_{-\infty}^{x_1+x_4}\alpha_x^{p+s-r}e^{-x^2}dx\right)e^{-\frac{1}{2}(x_1+x_4)^2}dx_4\right)\right\}$$

$$+\frac{\sqrt{2\pi}}{s+1}\left\{\sqrt{2\pi}\,(pI_{m-2}-{}_{p+s+1}I_{m-2})+\{(p-r)I_{p-r-1}-(p+s-r+1)I_{p+s-r}\}\right.$$

$$\times\int_{-\infty}^{+\infty}\alpha_{x_1}^{m-2}\frac{\sqrt{2\pi}}{r+1}\left[\alpha_{x_1+x_4}^{r+1}\right]_{x_4=0}^{\infty}e^{-x_1^2}dx_1$$

$$-\frac{p-r}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1}^{m-2}\left(\left[\frac{\sqrt{2\pi}}{r+1}\alpha_{x_1+x_4}^{r+1}\int_{-\infty}^{x_1+x_4}\alpha_x^{p-r-1}e^{-x^2}dx\right]_{x_4=0}^{\infty}\right.$$

$$\left.-\frac{\sqrt{2\pi}}{r+1}\int_0^{\infty}\alpha_{x_1+x_4}^{p}e^{-(x_1+x_4)^2}dx_4\right)e^{-x_1^2}dx_1$$

$$+\frac{p+s-r+1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\alpha_{x_1}^{m-2}\left(\left[\frac{\sqrt{2\pi}}{r+1}\alpha_{x_1+x_4}^{r+1}\int_{-\infty}^{x_1+x_4}\alpha_x^{p+s-r}e^{-x^2}dx\right]_{x_4=0}^{\infty}\right.$$

$$\left.\left.-\frac{\sqrt{2\pi}}{r+1}\int_0^{\infty}\alpha_{x_1+x_4}^{p+s+1}e^{-(x_1+x_4)^2}dx_4\right)e^{-x_1^2}dx_1\right\}\Bigg],$$

$$=\Phi'_{r,s}\left[\frac{2\pi\sqrt{2\pi}}{r+1}({}_{p+s+1}I_{m-2}-{}_{p+s-r}I_{m+r-1})+\frac{2\pi\sqrt{2\pi}\,s}{p-r+1}\left\{\left(\frac{1}{r+1}-\frac{1}{p+2}\right)I_{s-1}I_{m-2}\right.\right.$$

$$-\left(\frac{I_{m+r-1}}{r+1}-\frac{I_{p+m}}{p+2}\right)I_{s-1}-\left(\frac{1}{r+1}-\frac{1}{p+2}\right)I_{s-1}I_{m-2}+\frac{I_{s-1}}{r+1}I_{m+r-1}$$

$$\left.-\frac{1}{p+2}{}_{p+m}I_{s-1}-\frac{1}{p+2}{}_{p+s+1}I_{m-2}-\frac{1}{r+1}({}_{p+s-r}I_{m+r-1}-{}_{p+s+1}I_{m-2})\right\}$$

$$+\frac{2\pi\sqrt{2\pi}}{s+1}\left\{pI_{m-2}-{}_{p+s+1}I_{m-2}+\frac{1}{r+1}\right.$$

$$\left.\left.\times[(p+s-r+1)({}_{p+s-r}I_{m+r-1}-{}_{p+s+1}I_{m-2})-(p-r)({}_{p-r-1}I_{m+r-1}-{}_{p}I_{m-2})]\right\}\right],$$

$$=\frac{m-1}{2\pi}\left\{\frac{1}{p+1}\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\cdot{}_{p+s+1}I_{m-2}\right.$$

$$-\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\sum_{r=0}^{p}(-1)^r\frac{{}_pC_r}{r+1}{}_{p+s-r}I_{m+r-1}$$

$$-\frac{(-1)^p}{(p+1)(p+2)}\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\,s\,({}_{p+s+1}I_{m-2}-{}_{s-1}I_{p+m})$$

$$+\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\,s\sum_{r=0}^{p}(-1)^r{}_pC_r\frac{{}_{p+s+1}I_{m-2}-{}_{p+s-r}I_{m+r-1}}{(p-r+1)(r+1)}$$

$$-\frac{1}{p+1}\sum_{s=0}^{m-1}(-1)^s{}_{m-1}C_s\frac{p+s+2}{s+1}{}_{p+s+1}I_{m-2}$$

$$+\frac{1}{m}{}_pI_{m-2}+\sum_{s=0}^{m-1}(-1)^s\frac{{}_{m-1}C_s}{s+1}\sum_{r=0}^{p}(-1)^r\frac{{}_pC_r}{r+1}$$

$$\left.\times\left((p+s-r+1)\,{}_{p+s-r}I_{m+r-1}-(p-r)\,{}_{p-r-1}I_{m+r-1}\right)\right\},$$

or, second term of (42·1)

$$= \frac{m-1}{2\pi}\left\{\frac{1}{m}\,{}_pI_{m-2} + \frac{1}{(p+1)(p+2)}\sum(-1)^s\,{}_{m-1}C_s \times\left[\frac{(p+s+2)(s-p-1)}{s+1}\,{}_{p+s+1}I_{m-2} + (-1)^p\,{}_{s-1}I_{p+m}\right]\right.$$

$$-\frac{1}{m}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\,\frac{p-r}{r+1}\,{}_{p-r-1}I_{m+r-1}$$

$$\left.-\sum_{s=0}^{m-1}(-1)^s\,{}_{m-1}C_s\,\frac{1}{s+1}\sum_{r=0}^{p}(-1)^r\,{}_pC_r\,\frac{(p+s-r+1)(s+r-p)}{(p-r+1)(r+1)}\,{}_{p+s-r}I_{m+r-1}\right\} \quad \ldots\ldots(42{\cdot}3).$$

On substituting (42·2) and (42·3) in (42·1), we obtain

$$p_{11}' = \frac{n!}{8m!\,(m-1)!\,p!\,\pi}\left\{(m-1)\,{}_pI_{m-2}\right.$$

$$+\sum_{s=0}^{m-1}(-1)^s\,{}_{m-1}C_s\left[\frac{(p+s+2)(s-p-1)\,m(m-1)}{(p+1)(p+2)(s+1)}\,{}_{p+s+1}I_{m-2}\right.$$

$$\left.+(-1)^p\left(\frac{m(m-1)}{(p+1)(p+2)}-1\right)s_{-1}I_{p+m}\right]$$

$$+\sum_{r=0}^{p}(-1)^r\,{}_pC_r\left(\frac{r}{m}-\frac{m-1}{r+1}\right)(p-r)\,{}_{p-r-1}I_{m+r-1}$$

$$+\sum_{s=0}^{m-1}(-1)^s\,{}_{m-1}C_s\sum(-1)^r\,{}_pC_r\left[r\left(\frac{s}{p-r+1}-\frac{p-r}{s+1}\right)\right.$$

$$\left.\left.-\frac{m(m-1)(p+s-r+1)(s+r-p)}{(s+1)(p-r+1)(r+1)}\right]\,{}_{p+s-r}I_{m+r-1}\right\},$$

$$= \frac{n!}{8m!\,(m-1)!\,p!\,\pi}\left\{(m-1)\,{}_pI_{m-2}\right.$$

$$-\frac{1}{(p+1)(p+2)}\sum_{s=0}^{m-1}(-1)^s\,{}_{m-1}C_s\left[\frac{m(m-1)(p+s+2)(p-s+1)}{s+1}\,{}_{p+s+1}I_{m-2}\right.$$

$$\left.+(-1)^p(p-m+2)(p+m+1)\,s_{-1}I_{p+m}\right]$$

$$+\sum_{r=0}^{p}(-1)^r\,{}_pC_r\,\frac{(m+r)(r-m+1)(p-r)}{m(r+1)}\,{}_{p-r-1}I_{m+r-1}+\sum_{s=0}^{m-1}(-1)^s\,\frac{{}_{m-1}C_s}{s+1}$$

$$\left.\times\sum_{r=0}^{p}(-1)^r\,{}_pC_r\,\frac{(p+s-r+1)(p-s-r)\big(m(m-1)-r(r+1)\big)}{(r+1)(p-r+1)}\,{}_{p+s-r}I_{m+r-1}\right\} \quad \ldots\ldots(42).$$

On inserting the numerical results given in Table I into the general formulae (41) and (42) we have obtained the theoretical values for p_{11}', and hence from (39) and (40), using the already known values for σ_{q_1}, we have found the values of p_{11} and $\sigma_{q_1-q_2}$ given in Tables VI[a] and VI[b]. For $n = 2, 3, \ldots 6$ the true values of p_{12}, p_{13} and p_{22} were also calculated, and thence from (37·1) the values for μ_3, μ_4, β_1 and β_2 obtained. These also are given in Table VI[a].

TABLE VI[a].

Interquartile Distance. Theoretical Values of Frequency Constants.

n	Moment coefficients for $(q_1 - q_2)$							Product moment coefficients for q_1 and q_2			
	Mean	σ	μ_2	μ_3	μ_4	β_1	β_2	p_{11}	p_{12}	p_{13}	p_{22}
2	1·128 379	·852 50	·726 7605	·616 636	2·043 625	·9906	3·8692	·318 310	·077 079	·622 273	·696 036
3	1·692 569	·888 37	·789 1978	·452 962	2·046 879	·4174	3·2864	·164 868	·045 761	·266 003	·370 644
4	1·326 387	·585 96	·343 3546	·124 136	·512 203	·3807	4·3447	·164 161	·015 517	·162 498	·188 277
5	1·657 982	·621 30	·386 0098	·114 517	·473 267	·2280	3·1762	·108 925	·013 191	·097 032	·116 018
6	1·283 510	·589 35	·347 3352	·125 736	·399 518	·3773	3·3116	·105 910	·014 156	·086 727	·102 143
7	1·514 751	·594 55	·353 4939	—	—	—	—	·079 999	—	—	—
8	1·325 042	·473 30	·224 0117	—	—	—	—	·079 634	—	—	—
9	1·504 270	·482 14	·232 4616	—	—	—	—	·063 847	—	—	—
10	1·312 105	·468 96	·219 9208	—	—	—	—	·065 050	—	—	—
11	1·457 694	·476 41	·226 9620	—	—	—	—	·053 224	—	—	—
12	1·329 253	·406 38	·165 1476	—	—	—	—	·052 677	—	—	—

The validity of the assumptions made in reaching (37·2) was tested on the case $n = 6$. Using this equation, we find

$$p_{12} = -\cdot 007\,728 \text{ against true value of } +\cdot 014\,156,$$
$$p_{13} = +\cdot 091\,007 \quad \text{,,} \quad \text{,,} \quad +\cdot 086\,727,$$
$$p_{22} = +\cdot 101\,422 \quad \text{,,} \quad \text{,,} \quad +\cdot 102\,143,$$

and on substituting these first values into (37·1), we obtain

$$\beta_1 = \cdot 000\,74 \text{ against true value of } \cdot 377\,29,$$
$$\beta_2 = 2\cdot 991\,94 \quad \text{,,} \quad \text{,,} \quad 3\cdot 311\,61.$$

It is quite clear therefore that in the present problem the assumptions of linearity of regression and homoscedasticity between q_1 and q_2 will not provide approximations of any value to β_1 and β_2 in small samples.

(3) *Comparison of Theoretical and Experimental Sampling Results.*

The frequency constants of the experimental samplings are placed together in Table VII. The values of $c_{q_1-q_2}$, the ratio of $\sigma_{q_1-q_2}$ to $\tilde{\sigma}/\sqrt{2n}$ and of $r_{q_1 q_2}$ may be compared with the theoretical values in Table VI[b], where the experimental values are placed within squared brackets. The standard error of sampling is of course rather large, e.g. for $c_{q_1-q_2} = 2\cdot 22$, say, it is ·050 for 1000 samples ($N = 1000$), and ·070 for $N = 500$, while for $r_{q_1 q_2} = \cdot 33$, say, it is ·028 for $N = 1000$ and ·040 for $N = 500$.

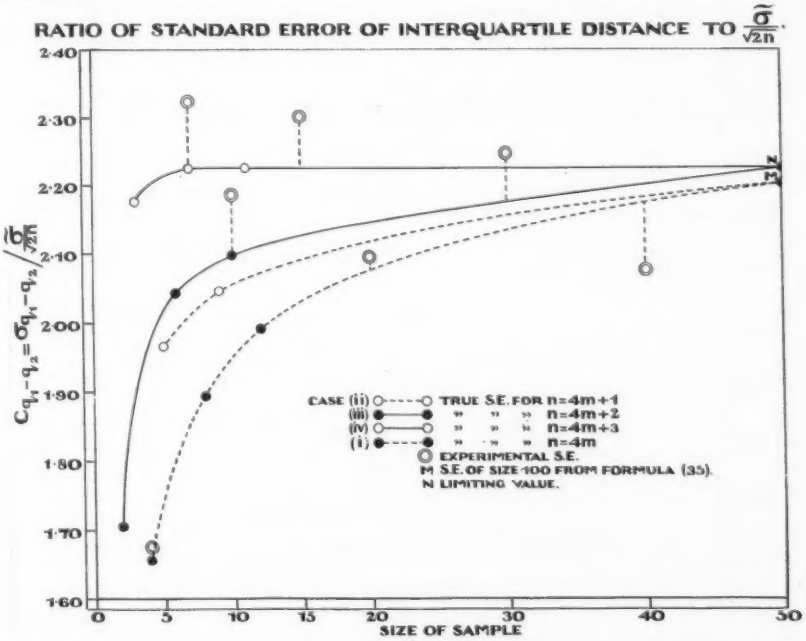

FIG. 6.

Mean Interquartile Distance. This is of course twice the mean deviation of a quartile, or the values are twice those given for $\bar{q}$ in Table IV[a]. The limiting value is $2 \times \cdot 67449 = 1 \cdot 34898$.

Standard Error of Interquartile Distance. The ratios $c_{q_1 - q_3}$ are shown in Table VI[b], separated out for the four different cases; the trend towards the limiting values is also represented in Fig. 6.

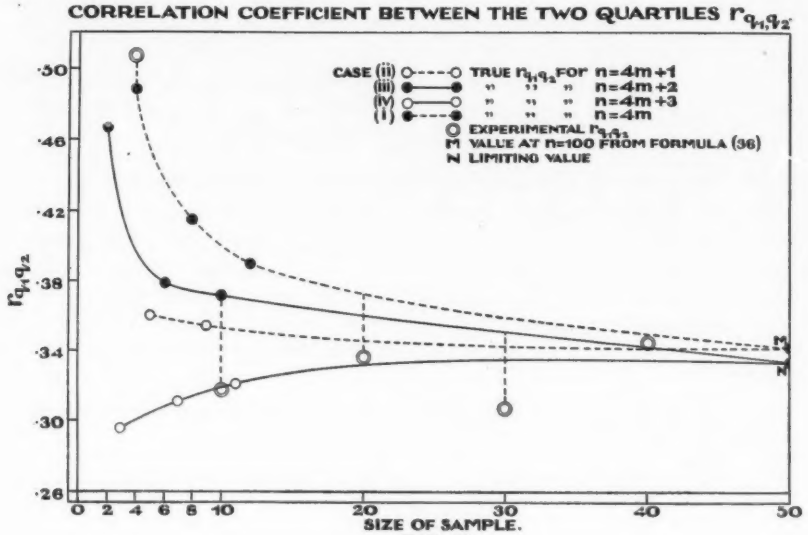

FIG. 7.

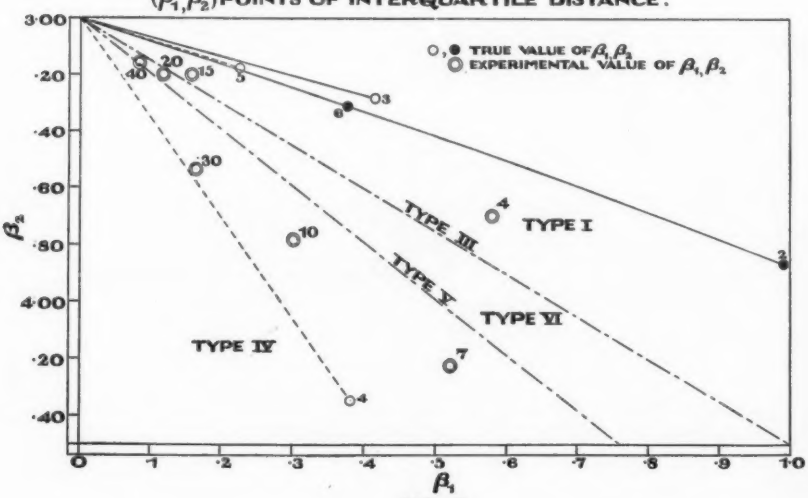

FIG. 8.

The Coefficient of Correlation $r_{q_1q_2}$. The values are again set out in Table VI[b], and the trend represented in Fig. 7. It appears that in the one case, $n = 4m + 2$, the correlation approaches the limiting value of $\frac{1}{3}$ from below.

Values of β_1 *and* β_2. The theoretical values up to $n = 6$ are shown in Table VI[a], and the experimental values in Table VII and Fig. 8. The distributions are quite definitely not normal; the theoretical limits for β_1 and β_2 are 0 and 3 as shown in (38·1) and (38·2), but the sampling results for $n = 20$, 30 and 40 suggest that this limit is not very rapidly approached. In the case $n = 20$, a normal curve and a Type III curve were fitted to the 1000 experimental values of $q_1 - q_2$. The test for goodness of fit gave $P = \cdot 245$ in the former and $P = \cdot 691$ in the latter case.

TABLE VI[b].

Interquartile Distance. Comparison of Theoretical with Experimental Values.

n	$c_{q_1-q_2}$. Ratio of standard error of $(q_1 - q_2)$ to $\tilde{\sigma}/\sqrt{2n}$				$r_{q_1q_2}$			
	Case (iii)	Case (iv)	Case (i)	Case (ii)	Case (iii)	Case (iv)	Case (i)	Case (ii)
2	1·7050				·4669			
3		2·1760				·2947		
4			1·6574 [1·6762]				·4888 [·5067]	
5				1·9647				·3608
6	2·0416				·3788			
7		2·2245 [2·3233]				·3116		
8			1·8932				·4155+	
9				2·0456				·3546
10	2·0972 [2·1853]				·3717 [·3176]			
11		2·2247				·3212		
12			1·9909				·3895−	
15		[2·3006]						
20			[2·0963]				[·3359]	
30	[2·2457]				[·3072]			
40			[2·0755]				[·3436]	
Large	2·2252*	2·2252*	2·2034†	2·2034†	·3333‡	·3333‡	·3377§	·3377§

* This is the limiting value given by equation (31).
† Value for $n = 100$ obtained from equations (34) and (35).
‡ Limiting value from equation (32).
§ Value for $n = 100$ from equations (34) and (36).
N.B. Experimental values are contained in square brackets.

(4) *Use of the Interquartile Distance to provide an Estimate of* $\tilde{\sigma}$.

If we write $Q = q_1 - q_2$ for the interquartile distance in a sample of n, and call $\tilde{\sigma} \times k_{q_1-q_2}$ the mean value of Q in repeated samples of n, then the numerical values of $\frac{1}{2}k_{q_1-q_2}$ up to $n = 13$ are the mean values, $\bar{q}$, given in Table IV[a]. It follows that

TABLE VII.

Interquartile Distance. Experimental Sampling Results.

Size of Sample, n (No. of Samples, N)	4 (1000)	7 (1000)	10 (1000)	15 (1000)	20 (1000)	30 (500)	40 (500)
Mean	1·347	1·528	1·320 3	1·425 8	1·332 25	1·342 4	1·340 9
μ_2	·351 2027	·385 5627	·238 7846	·176 4210	·109 8591	·084 0489	·053 8439
μ_3	·158 5950	·173 0788	·064 1599	·029 5833	·012 5018	·009 9148	·003 6496
μ_4	·456 7638	·628 4241	·215 7887	·099 5821	·038 6060	·024 9817	·009 1499
β_1	·580 6	·522 6	·302 4	·159 4	·117 9	·165 6	·085 3
β_2	3·703 2	4·227 3	3·784 6	3·199 5	3·198 8	3·536 4	3·156 1
$\sigma_{q_1-q_3}$	·592 6	·620 9	·488 7	·420 0	·331 5	·289 9	·232 0
$\sigma_{q_1-q_3} \Big/ \frac{\tilde{\sigma}}{\sqrt{2n}}$	1·676 2	2·323 3	2·185 3	2·300 6	2·096 3	2·245 7	2·075 5
$r_{q_1 q_3}$	·506 7		·317 6		·335 9	·307 2	·343 6

if we have available only a single small sample we may take $Q/k_{q_1-q_3}$ as an estimate of the unknown population standard deviation, $\tilde{\sigma}$. It will be an unbiassed estimate in the sense that its mean value in repeated sampling is $\tilde{\sigma}$. Then it is of interest to compare the reliability of this estimate with that obtained from the sample standard deviation, say s. For repeated samples from a normal population,

$$\text{mean } s = k_s \times \tilde{\sigma}, \quad \text{standard error of } s = \sigma_s = c_s \times \tilde{\sigma}/\sqrt{2n},$$

where the constants k_s and c_s have been tabled for certain values of n from 2 to 100*. Hence our problem is to compare the standard errors of these two estimates of $\tilde{\sigma}$,

$$(a)\ E_2 = Q/k_{q_1-q_3} \text{ and } (b)\ E_1 = s/k_s.$$

(*a*) The standard error of $E_2 = \dfrac{\sigma_{q_1-q_3}}{k_{q_1-q_3}} = \dfrac{c_{q_1-q_3}}{k_{q_1-q_3}} \times \dfrac{\tilde{\sigma}}{\sqrt{2n}} = \theta_2 \times \dfrac{\tilde{\sigma}}{\sqrt{2n}}$, where the values of $k_{q_1-q_3}$ are given in the second column of Table VI[a] and of Table VI[b].

(*b*) The standard error of $E_1 = \dfrac{\sigma_s}{k_s} = \dfrac{c_s}{k_s} \times \dfrac{\tilde{\sigma}}{\sqrt{2n}} = \theta_1 \times \dfrac{\tilde{\sigma}}{\sqrt{2n}}$.

The values of θ_2 and θ_1 and their ratio are given in Table VIII for values of n from 2 to 12, and also for the limiting case of very large samples. The ratio θ_2/θ_1 is always greater than unity and tends in the limit to 1·6495. Hence it follows that the standard deviation (or variance) may be used to provide a definitely more reliable estimate of $\tilde{\sigma}$ than can be obtained from the sample interquartile distance. In exactly the same way the sample mean provides a more reliable estimate of the population mean than does the sample median. But in this case the limiting ratio of the standard errors is nearer unity, namely 1·2533. For populations which are not normal this result will not of course hold, nor will the numerical multipliers be

* *Biometrika*, Vol. x. p. 528 and Vol. xi. p. 279.

the same, and for extremely leptokurtic symmetrical populations it is possible that more reliable estimates of both the mean and the standard deviation of the population could be obtained from the sample median and interquartile distance, rather than from the sample mean and standard deviation *.

TABLE VIII.

The Standard Errors of Estimates of $\tilde{\sigma}$.

n	θ_2 (S.E. of E_2) $\Big/ \frac{\tilde{\sigma}}{\sqrt{2n}}$	θ_1 (S.E. of E_1) $\Big/ \frac{\tilde{\sigma}}{\sqrt{2n}}$	(S.E. of E_2)/(S.E. of E_1)			
			$4m$	$4m+1$	$4m+2$	$4m+3$
2	1·5110	1·5110			1·0000	1·0042
3	1·2856	1·2804				
4	1·2496	1·1936	1·0469			
5	1·1850	1·1480		1·0322		
6	1·5906	1·1196			1·4207	
7	1·4686	1·1004				1·3344
8	1·4288	1·0865	1·3125			
9	1·3599	1·0760		1·2638		
10	1·5983	1·0678			1·4968	
11	1·5262	1·0611				1·4383
12	1·4978	1·0558	1·4186			
∞	1·6495	1·0000	1·6495	1·6495	1·6495	1·6495

IV. *Summary and Conclusions.*

The object of this paper has been to examine the sampling variation of the median, quartiles and interquartile distance in samples from a normal population, and to show how their distributions in the case of very small samples tend towards the limiting forms.

The sampling distribution of the median is very nearly normal in all cases; it provides a somewhat less reliable estimate of the position of the population mean than the sample mean does. In this connection the following comparison of three "central estimates" is of interest. If x_1 be the lowest and x_2 the highest variate value in a sample, $\frac{1}{2}(x_1+x_2)$ is the "centre" or mid-point between extremes. Then if $\tilde{\sigma}/\sqrt{n}$, the standard error of the mean, be taken as unity, we find

Size of sample	1	2	3	4	5	6	10	20	∞
S. E. of mean	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
S. E. of median	1·000	1·000	1·160	1·092	1·198	1·135	1·177	1·214	1·253
S. E. of centre	1·000	1·000	1·042	1·092	1·142	1·190	1·362	1·691	[7·858]†

The mean is definitely the most reliable; for very small samples there is little to choose in accuracy between the median and the centre. For large samples however the centre becomes relatively increasingly unreliable, while the standard error

* See paper by E. S. Pearson and N. K. Adyanthaya. *Biometrika*, Vol. XXI. p. 358.

† Approximate value for $n=1000$.

of the median has almost attained its limiting value by the time $n = 20$. It must also be remembered that the standard error of the mean is invariably $\tilde{\sigma}/\sqrt{n}$ whatever be the form of population distribution; the numerical factors given above for the median and centre will change with the population form. Further if this be asymmetrical the mean position of median and centre in repeated samples no longer lies at the population mean.

The distribution of the quartile is not so closely normal as that of the median, and here there are four cases to be considered. The results shown in Fig. 5, however, suggest that by the time $n = 10$ the assumption of normality would not lead to serious error. The standard error of the quartile is somewhat greater than that of the median, tending to a limiting value of $1{\cdot}3626\ \tilde{\sigma}/\sqrt{n}$ as against $1{\cdot}2533\ \tilde{\sigma}/\sqrt{n}$. The curves representing the four cases are shown in Fig. 4.

The distribution of the interquartile distance is definitely skew in small samples; the longer tail of the distribution being towards the larger values of the measure. The experimental results even at $n = 40$ still show definite asymmetry. But the distribution tends to the normal slowly as n increases. A method of estimating the population standard deviation, $\tilde{\sigma}$, from the interquartile distance, Q, has been considered; it consists in multiplying Q by a numerical factor to give an estimate whose mean value in repeated samples would be $\tilde{\sigma}$. It is not, however, as reliable an estimate as that which can be obtained from the sample standard deviation, s, and in large samples its standard error tends to be $1{\cdot}6495 \times$ S. E. of s.

In the paper cited above K. Pearson has considered methods of estimating $\tilde{\sigma}$ from other percentile distances, and has shown* that in the limit a better estimate can be obtained from taking the range between the ordinates at $\frac{1}{14}$ of the area from either end, or about the 7th and 93rd percentile. But even in this case the method is less reliable than that of using s, being 24 °/₀ greater as compared with the 65 °/₀ at the quartiles.

The position of the median and the quartiles can be found in small samples with such rapidity, that in certain cases the time saved may be felt to compensate for the accuracy lost. The results of this paper have served to indicate the nature of this loss in accuracy.

* *Biometrika*, Vol. XIII. p. 119.

APPENDIX TO A PAPER BY PROFESSOR TOKISHIGE HOJO.

On the Standard Error of the Median to a Third Approximation when the Median is found from a Sample of size $n=2p+1$, and the Parent Population is normal and of Standard Deviation σ.

By KARL PEARSON.

Let x be the variate measured from the mean of the normal population and $x'=x/\sigma$. Further let a_x' be given by

$$a_x' = \frac{1}{\sqrt{2\pi}\,\sigma}\int_0^x e^{-\frac{1}{2}x^2/\sigma^2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{x'} e^{-\frac{1}{2}x'^2}\,dx' = \alpha'_{x'}.$$

We require as a preliminary to expand a_x' in terms of x'. We have

$$a_x' = (\alpha'_{x'})_0 + \left(\frac{d\alpha'_{x'}}{dx'}\right)_0 x' + \frac{1}{2!}\left(\frac{d^2\alpha'_{x'}}{dx'^2}\right)_0 x'^2 + \text{etc.}$$

Now $\dfrac{d\alpha'_{x'}}{dx'} = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x'^2}$, and $\left(\dfrac{d\alpha'_{x'}}{dx'}\right)_0 = \dfrac{1}{\sqrt{2\pi}} = z_0$, say,

$$\frac{d^2\alpha'_{x'}}{dx'^2} = -x'\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x'^2}, \text{ and } \frac{d^2\alpha'_{x'}}{dx'^2} = 0,$$

$$\frac{d^3\alpha'_{x'}}{dx'^3} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x'^2}(x'^2-1), \text{ and } \left(\frac{d^3\alpha'_{x'}}{dx'^3}\right)_0 = -z_0,$$

$$\frac{d^4\alpha'_{x'}}{dx'^4} = -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x'^2}x'(x'^2-3), \text{ and } \left(\frac{d^4\alpha'_{x'}}{dx'^4}\right)_0 = 0,$$

$$\frac{d^5\alpha'_{x'}}{dx'^5} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x'^2}(x'^4-6x'^2+3), \text{ and } \left(\frac{d^5\alpha'_{x'}}{dx'^5}\right)_0 = 3z_0,$$

$$\frac{d^6\alpha'_{x'}}{dx'^6} = -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x'^2}x'(x'^4-10x'^2+15), \text{ and } \left(\frac{d^6\alpha'_{x'}}{dx'^6}\right)_0 = 0.$$

Thus we have

$$\alpha'_{x'} = z_0x'(1-\tfrac{1}{6}x'^2+\tfrac{1}{40}x'^4).$$

Squaring this we have to the same order of approximation

$$\left(\frac{\alpha'_{x'}}{z_0}\right)^2 = x'^2(1-\tfrac{1}{3}x'^2+\tfrac{7}{90}x'^4),$$

or

$$x'^2 = \left(\frac{\alpha'_{x'}}{z_0}\right)^2(1+\tfrac{1}{3}x'^2+\tfrac{1}{30}x'^4) \quad \text{.....(i).}$$

Hence finally on inversion

$$x'^2 = \left(\frac{\alpha'_{x'}}{z_0}\right)^2 + \frac{1}{3}\left(\frac{\alpha'_{x'}}{z_0}\right)^4 + \frac{13}{90}\left(\frac{\alpha'_{x'}}{z_0}\right)^6 \quad \text{.....(ii).}$$

If we are dealing with a sample of fair size the bulk of the values will cluster round the parent-population mean where $\alpha'_{x'}$ will be small, and accordingly the above will give a reasonable approximation for x' in terms of $\alpha'_{x'}$.

Now if σ_m be the standard error of the median, we have

$$\sigma_m{}^2 = \frac{2p+1!}{(p!)^2} \times \int_{-\infty}^{+\infty} a_x{}^p (1-a_x)^p x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x^2/\sigma^2)} dx,$$

where

$$a_x = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x^2/\sigma^2)} dx.$$

Now $a_x = 0{\cdot}5 + a'_{x'}$ and replacing x by x' and a_x by $a'_{x'}$ we have

$$\sigma_m{}^2 = \sigma^2 \frac{2p+1!}{(p!)^2} \times 2 \int_0^{\frac{1}{2}} (\tfrac{1}{4} - a'_{x'}{}^2)^p x'^2 da'_{x'}$$

$$= \sigma^2 \frac{2p+1!}{2^{2p-1}(p!)^2} \int_0^{\frac{1}{2}} (1 - 4a'_{x'}{}^2)^p \left(\left(\frac{a'_{x'}}{z_0} \right)^2 + \frac{1}{3} \left(\frac{a'_{x'}}{z_0} \right)^4 + \frac{13}{90} \left(\frac{a'_{x'}}{z_0} \right)^6 \right) da'_{x'},$$

making use of the value for x'^2 given in (ii) above.

Now put $4a'_{x'}{}^2 = u$, then we have

$$\sigma^2{}_m = \frac{\sigma^2 \Gamma(2p+2)}{2^{2p+2}\Gamma^2(p+1)} \frac{1}{2z_0{}^2} \int_0^1 (1-u)^p \left(u^{\frac{1}{2}} + \frac{1}{12} \frac{u^{\frac{3}{2}}}{z_0{}^2} + \frac{13}{1440} \frac{u^{\frac{5}{2}}}{z_0{}^4} \right) du$$

$$= \frac{\sigma^2}{2z_0{}^2} \frac{\Gamma(2p+2)}{2^{2p+2}\Gamma(p+1)} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(p+\frac{5}{2})} + \frac{1}{12} \frac{\Gamma(\frac{5}{2})}{z_0{}^2\Gamma(p+\frac{7}{2})} + \frac{13}{1440} \frac{\Gamma(\frac{7}{2})}{z_0{}^4\Gamma(p+\frac{9}{2})} \right)$$

$$= \frac{\sigma^2}{2z_0{}^2} \frac{\Gamma(2p+2)\sqrt{\pi}}{2^{2p+3}\Gamma(p+1)\Gamma(p+\frac{5}{2})} \left(1 + \frac{1}{4z_0{}^2} \frac{1}{2^{p+5}} + \frac{13}{96z_0{}^4} \frac{1}{(2p+5)(2p+7)} \right)$$

.....(iii).

We will now proceed to evaluate the separate components of $\sigma_m{}^2$ on the assumption that p is considerable.

First

$$u = \frac{\Gamma(2p+2)\sqrt{\pi}}{2^{2p+3}\Gamma(p+1)\Gamma(p+\frac{5}{2})}.$$

We use Stirling's Theorem

$$\Gamma(s+1) = \sqrt{2\pi}\, s^{s+\frac{1}{2}} e^{-s} \chi_s,$$

where

$$\chi_s = 1 + \frac{1}{12s} + \frac{1}{288s^2} - \text{etc.},$$

and find

$$u = \frac{1}{4p} \frac{\left(1 + \frac{1}{2p}\right)^{2p+\frac{3}{2}}}{\left(1 + \frac{3}{2p}\right)^{p+2}} e^{-\frac{1}{2}} \frac{\chi_{2p+1}}{\chi_p \chi_{p+\frac{3}{2}}}.$$

Now

$$\frac{\chi_{2p+1}}{\chi_p \chi_{p+\frac{3}{2}}} = \frac{\left(1 + \frac{1}{24p} - \frac{23}{1152p^2} + \ldots\right)}{\left(1 + \frac{1}{12p} + \frac{1}{288p^2} + \ldots\right)\left(1 + \frac{1}{12p} - \frac{35}{388p^2} + \ldots\right)}$$

$$= 1 - \frac{1}{8p} + \frac{43}{384p^2} + \ldots.$$

Again
$$\log_e \frac{\left(1+\frac{1}{2p}\right)^{2p+\frac{1}{2}}}{\left(1+\frac{3}{2p}\right)^{p+2} e^{-\frac{1}{2}}} = -\frac{11}{8p} + \frac{49}{48p^2},$$

or
$$\frac{\left(1+\frac{1}{2p}\right)^{2p+\frac{1}{2}}}{\left(1+\frac{3}{2p}\right)^{p+2} e^{-\frac{1}{2}}} = 1 - \frac{11}{8p} + \frac{755}{384p^2} + \ldots.$$

Accordingly
$$u = \frac{1}{4p}\left(1 - \frac{3}{2p} + \frac{9}{4p^2} + \ldots\right).$$

Finally we have

$$\sigma_m{}^2 = \frac{\sigma^2}{2z_0{}^2}\frac{1}{4p}\left(1 - \frac{3}{2p} + \frac{9}{4p^2} + \ldots\right)\left(1 + \frac{1}{4z_0{}^2(2p+5)} + \frac{13}{96z_0{}^4(2p+5)(2p+7)} + \ldots\right).$$

Now $2p = n-1$, and $z_0 = \cdot 398{,}9423$ and we will express σ_m in terms of $\frac{\sigma}{\sqrt{n}}$ and inverse powers of n.

$$\frac{\sigma_m}{\sigma/\sqrt{n}} = 1\cdot 253{,}314\sqrt{\frac{n}{n-1}}\left(1 - \frac{3}{n} + \frac{6}{n^2}\ldots\right)^{\frac{1}{2}}\left(1 + \frac{1\cdot 570{,}7960}{n} - \frac{\cdot 937{,}1511}{n^2}\right)^{\frac{1}{2}}$$

$$= 1\cdot 253{,}314\left(1 + \frac{1}{2n} + \frac{3}{8n^2}\right)\left(1 - \frac{3}{2n} + \frac{15}{8n^2}\ldots\right)\left(1 + \frac{\cdot 785{,}3980}{n} - \frac{\cdot 777{,}0006}{n^2}\right)$$

$$= 1\cdot 253{,}314\left(1 - \frac{1}{n} + \frac{1\cdot 5}{n^2}\right)\left(1 + \frac{\cdot 785{,}3980}{n} - \frac{\cdot 777{,}0006}{n^2}\right)$$

$$= 1\cdot 253{,}314\left(1 - \frac{\cdot 214{,}6020}{n} - \frac{\cdot 062{,}3980}{n^2}\right).$$

Thus

$$\frac{\sigma_m}{\sigma/\sqrt{n}} = 1\cdot 253{,}314 - \frac{\cdot 268{,}964}{n} - \frac{\cdot 078{,}205}{n^2} + \text{etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv).}$$

The curve obtained by Professor Hojo, who fitted a cubic in $\frac{1}{n}$ from the values $n = 3, 5, 7, 9$ and the limiting value, by the method of moments, was

$$\frac{\sigma_m}{\sigma/\sqrt{n}} = 1\cdot 2533 - \frac{\cdot 2653}{n} - \frac{\cdot 0699}{n^2} + \frac{\cdot 0822}{n^3}.$$

It is possible that the first three terms of our expression (iv) when n is $=$ or >11 give as good results as Hojo's cubic based on the low values of n.

P.S. Since the above note was sent to press Equation (iii) has been extended to much higher powers of $\alpha'_{x'}/z_0$ and many additional results obtained which will appear in a later paper.

ON THE MEAN CHARACTER AND VARIANCE OF A RANKED INDIVIDUAL, AND ON THE MEAN AND VARIANCE OF THE INTERVALS BETWEEN RANKED INDIVIDUALS.

By KARL PEARSON with the assistance of MARGARET V. PEARSON.

Part I. Symmetrical Distributions (Normal and Rectangular).

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(i) *Introduction.*

In 1920 I published a study* of the characters of ranked individuals on the assumption that we had to deal with large samples. Still earlier (1901) I had dealt with a like matter, namely in discussing "Galton's Problem,"†—the question of suitable values for first, second and third prizes in the competition of a given number of individuals. While the treatment of a considerable population as a continuum as far as any character is concerned has many analytical advantages, there is a fascination, as well as certain practical uses, in thinking of such a population as a discontinuum with certain intervals between its members regulated according to the laws of chance. In actual experience, of course, every population is of this character. Given a definite problem concerning an observed population the continuous or discontinuous conception of it may be the more serviceable having regard to the object in view. Sometimes as in judging whether an outlying observation should be rejected both may be applicable‡.

* *Biometrika*, Vol. XIII. pp. 113—132. † *Ibid.* Vol. I. pp. 390—399.

‡ See Pearson: *Tables for Statisticians and Biometricians*, Part II, Introduction, pp. cv—cx.

In considering the characters of ranked individuals, and their intervals, the size of the sample plays an important part, and it soon became obvious that it is essential to discover how soon, for practical purposes in the case of ranking we may treat a sample as "large," that is to say, how soon may we adopt the asymptotic values for our means, variances and correlations.

Professor Hojo, working in the Biometric Laboratory, started at my suggestion to determine experimentally how soon the asymptotic values would be available, but unfortunately it was found that the errors of random sampling, even for sets of 500 or 1000 samples, were too great to obtain in this manner any definite appreciation of what might be the ranking constants in the case of very small samples. At the suggestion of Dr Egon S. Pearson, Professor Hojo changed his line of attack and was able to obtain theoretically the ranking constants for characters distributed normally in the case of samples of sizes two to twelve. He succeeded in achieving this by throwing back his results on the quadrature of certain integrals. Unfortunately the number of these integrals increased with the size of the sample, and the labour both algebraic and arithmetical became excessive even with samples of twelve. Granted, however, that the quadratures are accurately made, as I think they undoubtedly were in Professor Hojo's case, his method provides many of the ranking constants for these small samples. Unfortunately it did not (i) effectively link up the small sample with the asymptotic ranking constants, and (ii) it did not enable one to determine by a more or less definite process the absolute or relative ranking constants in a sample of any size.

The present paper endeavours to supplement in the above respects that of Professor Hojo. My method for very small samples in the case of normal distribution is not as accurate as his. But for samples about where he leaves off the values given nearly coincide, and we can accordingly measure the approach to the asymptotic values. In the case of a normal distribution, however, the method being approximative, we shall not obtain very good results even for large samples, when we deal with ranks close to the beginning or end of a series. Caution must be applied if the formulae be used for ranks beyond the 80th or below the 20th percentiles.

The method adopted is perfectly general and can be applied with absolute accuracy for any frequency curve in which the abscissa can be expressed in terms of the area of the frequency curve which stands upon that abscissa. Thus accurate results can be obtained at once for a rectangular distribution of frequency. The difficulty about the normal distribution is that the abscissa can only be expanded in terms of the area with an adequate degree of approximation for a limited range of the abscissa. Beyond this range, however, the frequency will in most cases be so insignificant, that the error in the abscissa is not of great importance. It does become of importance, however, if we are dealing with individuals whose rank carries them beyond the above limited range.

(ii) *Deduction of Mean Character of any Rank from a Table of Mean Ranges.*

Let ${}_n\bar{x}_q$ be the mean position of the qth individual in ranking in samples of n taken from an indefinitely large normal population. Then if α_x be the proportional area up to this individual

$$ {}_n\bar{x}_q = \frac{n!}{(q-1)!\,(n-q)!}\int_0^1 \alpha_x^{q-1}(1-\alpha_x)^{n-q}\,x\,d\alpha_x \quad \text{.....(i),} $$

where

$$ \alpha_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\,dx, $$

the standard deviation (σ) of the parent population being taken as unity, so that the left-hand side of (i) must be given the value ${}_n\bar{x}_q/\sigma$, if this be not the case.

If q be the last individual, then $q=n$, and we have

$$ {}_n\bar{x}_n = n\int_0^1 \alpha_x^{n-1}\,x\,d\alpha \quad \text{.....(ii).} $$

Thus the integral

$$ \int_0^1 \alpha_x^{s-1}\,x\,d\alpha_x = \frac{s\bar{x}_s}{s} = \frac{u_s}{s}, \text{ say.} $$

Returning to (i) we can expand the binomial under the integral sign and find

$$ \begin{aligned} {}_n\bar{x}_q &= \frac{n!}{(q-1)!\,(n-q)!}\sum_{t=0}^{n-q}\frac{(n-q)!\,(-1)^t}{t!\,(n-q-t)!}\int_0^1 \alpha_x^{q-1}\alpha_x^{t}\,x\,d\alpha \\ &= \frac{n!}{(q-1)!}\sum_{t=0}^{n-q}\frac{(-1)^t}{t!\,(n-q-t)!}\frac{u_{q+t-1}}{q+t-1} \\ &= \frac{n!}{(q-1)!\,(n-q)!}\left(\frac{u_q}{q} - \frac{n-q}{1!}\frac{u_{q+1}}{q+1} + \frac{(n-q)(n-q-1)}{2!}\frac{u_{q+2}}{q+2}\right. \\ &\qquad \left. - \frac{(n-q)(u-q-1)(n-q-2)}{3!}\frac{u_{q+3}}{q+3} + \ldots\right) \quad \text{.....(iii).} \end{aligned} $$

This may be written, if we put $w_s = 2u_s$ = mean range from first to last individual

$$ \begin{aligned} {}_n\bar{x}_q &= \frac{\frac{1}{2}(n)!}{(q-1)!\,(n-q)!}\left(\frac{w_q}{q} - \frac{n-q}{1!}\frac{w_{q+1}}{q+1} + \frac{(n-q)(u-q-1)}{2!}\frac{w_{q+2}}{q+2}\right. \\ &\qquad \left. - \frac{(u-q)(u-q-1)(u-q-2)}{3!}\frac{w_{q+3}}{q+3} + \ldots\right) \quad \text{.....(iii)}^{bis}. \end{aligned} $$

Now w_q has been tabled by Tippett from $q=0$ to 1000*. Unfortunately the published table only goes to five decimal places, and when u becomes large the multiplying factors become large, and the process becomes laborious and is liable to be inexact. It is easily applied to the low values of u dealt with by Professor Hojo. I owe the accompanying table to *six* decimals to the kindness of Mr Tippett, who sent me his MS. table of Mean Ranges.

* *Biometrika*, Vol. XVII. pp. 386—7. Also *Tables for Statisticians and Biometricians*, Part II, Table XXII, p. 165 and p. cx.

Table of Mean Ranges to Six Decimal Places (Tippett).

Size of Sample	Mean Range	Size of Sample	Mean Range
2	1·128,379	17	3·587,885−
3	1·692,569	18	3·640,064
4	2·058,750	19	3·688,963
5	2·325,930	20	3·734,950
6	2·534,413	21	3·778,336
7	2·704,357	22	3·819,385−
8	2·847,201	23	3·858,324
9	2·970,026	24	3·895,348
10	3·077,595+	25	3·930,629
11	3·172,873	26	3·964,316
12	3·258,455+	27	3·996,539
13	3·335,981	28	4·027,414
14	3·406,763	29	4·057,044
15	3·471,826	30	4·085,522
16	3·531,983	31	4·112,929

Let us take first the case of $n = 4m + 2$, and consider the quartile means, $q = 3m + 2$, in samples. Professor Hojo has taken the cases of 2, 6, 10 or $m = 0, 1, 2$. Thus we have three cases for (n, q) or (2, 2), (6, 5) and (10, 8), and accordingly

$$_2\bar{x}_1 = \frac{w_2}{2} = \cdot 564{,}190 \text{ as against Hojo's } \cdot 564{,}19,$$

$$_6c_5 = 15\left(\frac{w_5}{5} - \frac{w_6}{6}\right) = \cdot 641{,}753 \text{ as against Hojo's } \cdot 641{,}76,$$

$$_{10}\bar{x}_8 = 180\left(\frac{w_8}{8} - \frac{2w_9}{9} + \frac{w_{10}}{10}\right) = \cdot 656{,}072 \text{ as against Hojo's } \cdot 656{,}05.$$

It is clear that already at 10, we should require Tippett's Table to more figures than we yet have owing to the size of the multipliers.

Next let us take $n = 4m + 3$, and $q = 3m + 3$. We have for samples (3, 3), (7, 6) and (11, 9),

$$_3\bar{x}_3 = \tfrac{3}{2}\left(\frac{w_3}{3}\right) = \cdot 846{,}285 \text{ as against Hojo's } \cdot 846{,}284,$$

$$_7\bar{x}_6 = 21\left(\frac{w_6}{6} - \frac{w_7}{7}\right) = \cdot 757{,}396 \text{ as against Hojo's } \cdot 757{,}375,$$

$$_{11}\bar{x}_9 = \tfrac{4\,9\,5}{2}\left(\frac{w_9}{9} - \frac{2w_{10}}{10} + \frac{w_{11}}{11}\right) = \cdot 728{,}863 \text{ as against Hojo's } \cdot 728{,}847.$$

We now turn to the rather more complicated cases of $n = 4m$ and $n = 4m + 1$, wherein no individual occupies the quartile position. First $n = 4m$; for this we require by the usual definition $_{4m}\bar{x}_{3m+\frac{1}{2}} = \frac{1}{2}\,(_{4m}\bar{x}_{3m} + {}_{4m}\bar{x}_{3m+1})$ or for samples of 4, 8, 12 respectively

$$_4\bar{x}_{3\cdot5} = \tfrac{1}{2}\,(_4\bar{x}_3 + {}_4\bar{x}_4) = \tfrac{1}{2}\left(6\left(\frac{w_3}{3} - \frac{w_4}{4}\right) + 2\,\frac{w_4}{4}\right)$$

$$= \tfrac{1}{2}\,(2w_3 - w_4) = \cdot 663{,}194 \text{ as against Hojo's } \cdot 663{,}194.$$

Again $${}_8\bar{x}_{6\cdot5} = \tfrac{1}{2}({}_8\bar{x}_6 + {}_8\bar{x}_7) = \tfrac{1}{2}\left(84\left(\frac{w_6}{6} - \frac{2w_7}{7} + \frac{w_8}{8}\right) + 28\left(\frac{w_7}{7} - \frac{w_8}{8}\right)\right)$$

$$= 7w_6 - 10w_7 + \tfrac{7}{2}w_8 = \cdot662{,}524 \text{ as against Hojo's } \cdot662{,}521.$$

Further

$${}_{12}\bar{x}_{9\cdot5} = \tfrac{1}{2}({}_{12}\bar{x}_9 + {}_{12}\bar{x}_{10}) = \tfrac{1}{2}\left(990\left(\frac{w_9}{9} - \frac{3w_{10}}{10} + \frac{3w_{11}}{11} - \frac{w_{12}}{12}\right)\right.$$

$$\left. + 330\left(\frac{w_{10}}{10} - \frac{2w_{11}}{11} + \frac{w_{12}}{12}\right)\right)$$

$$= \tfrac{1}{2}(110w_9 - 264w_{10} + 210w_{11} - 55w_{12}) = \cdot664{,}922$$

as against Hojo's $\cdot664{,}627$.

Next considering $n = 4m + 1$, we have

$${}_{4m+1}\bar{x}_{3m+1\cdot5} = \tfrac{1}{2}({}_{4m+1}\bar{x}_{3m+1} + {}_{4m+1}\bar{x}_{3m+2}),$$

and taking $m = 0$, 1, 2 and 3, obtain the mean quartile for samples of 1, 5, 9 and 13, and we need only consider the last three,

$${}_5\bar{x}_{4\cdot5} = \tfrac{1}{2}({}_5\bar{x}_4 + {}_5\bar{x}_5) = \tfrac{1}{2}\left(10\left(\frac{w_4}{4} - \frac{w_5}{5}\right) + \tfrac{5}{2}\frac{w_5}{5}\right) = \tfrac{1}{4}(5w_4 - 3w_4)$$

$$= \cdot828{,}990 \text{ as against Hojo's } \cdot828{,}991.$$

$${}_9\bar{x}_{7\cdot5} = \tfrac{1}{2}({}_9\bar{x}_7 + {}_9\bar{x}_8) = \tfrac{1}{2}\left(126\left(\frac{w_7}{7} - \frac{2w_8}{8} + \frac{w_9}{9}\right) + 36\left(\frac{w_8}{8} - \frac{w_9}{9}\right)\right)$$

$$= \tfrac{1}{2}(18w_7 - 27w_8 + 10w_9) = \cdot752{,}130 \text{ as against Hojo's } \cdot752{,}135.$$

Lastly

$${}_{13}\bar{x}_{10\cdot5} = \tfrac{1}{2}({}_{13}\bar{x}_{10} + {}_{13}\bar{x}_{11})$$

$$= \tfrac{1}{2}\left(1430\left(\frac{w_{10}}{10} - \frac{3w_{11}}{11} + \frac{3w_{12}}{12} - \frac{w_{13}}{13}\right) + 429\left(\frac{w_{11}}{11} - \frac{2w_{12}}{12} + \frac{w_{13}}{13}\right)\right)$$

$$= \tfrac{1}{2}(143w_{10} - 351w_{11} + 286w_{12} - 77w_{13})$$

$$= \cdot726{,}193 \text{ as against Hojo's } \cdot726{,}366.$$

It is thus clear that a table of mean ranges would enable us to obtain fairly easily the mean quartile for small samples, perhaps even up to $n = 20$. It is equally conspicuous, however, that the table would require to be carried to far more figures than are yet published, in order to obtain these means to five or six figure accuracy even for $n > 10$.

The results given above serve, however, to confirm in a general way Hojo's results for the quartile means of small samples.

Both methods become very laborious as n increases in size. Accordingly I have sought for a method, which, without the use of quadratures or a table similar to Tippett's, would enable us to find the mean and standard deviation of any rank in sampling, and, if needful, the higher moments with accuracy adequate for most practical purposes.

(iii) *To find the Mean of any Rank in Samples.*

Let the size of the sample be n, and let q be the rank and x the deviate measured from the population mean of the individual under consideration. Then if we suppose the population sampled to be normal, we have for the mean value $\bar{x}_q$ of the variate of the qth individual

$$\left.\begin{aligned} {}_n\bar{x}_q &= \frac{n!}{(q-1)!\,(n-q)!}\int_0^1 \alpha_x^{q-1}(1-\alpha_x)^{n-q}\,x\,d\alpha_x \\ \text{where}\qquad \alpha_x &= \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx \end{aligned}\right\} \quad \text{.....(iv),}$$

and ${}_n\bar{x}_q$ is measured from the mean of the sampled population.

Similarly the second moment about the same origin is

$${}_n\mu'_{2,q} = \frac{n!}{(q-1)!\,(n-q)!}\int_0^1 \alpha_x^{q-1}(1-\alpha_x)^{n-q}\,x^2 d\alpha_x \quad \text{.....(v),}$$

and the standard deviation ${}_n\sigma_q$ will be given by

$${}_n\sigma_q^{\,2} = {}_n\mu'_{2,q} - {}_n\bar{x}_q^{\,2} \text{.....(vi).}$$

Further, any moment ${}_n\mu'_{s,q}$ about the same origin will be given by

$${}_n\mu'_{s,q} = \frac{n!}{(q-1)!\,(n-q)!}\int_0^1 \alpha_x^{q-1}(1-\alpha_x)^{n-q}\,x^s d\alpha_x \text{.....(vii),}$$

and the moments about the mean, ${}_n\bar{x}_q$, can be obtained by the usual transference equations.

From Equations (iv), (v), (vi) and (vii) we note that if x could be expressed as a function of α_x, all the above expressions would reduce to complete B-functions, and accordingly to complete Γ-functions; thus the determination of the constants would reduce to straightforward arithmetic. Now we cannot hope to expand x in a series of integer positive powers of α by Taylor's Theorem, because of the difficulty of convergency and the fact that x must be infinite when $\alpha_x = 0$. We may notice, however, two points: first, when n is large and q is not nearly equal to n, the frequency of a value of α_x lying between α_x and $\alpha_x + \delta\alpha_x$, which is

$$\frac{n!}{(q-1)!\,(n-q)!}\alpha_x^{q-1}(1-\alpha_x)^{n-q}\,\delta\alpha_x,$$

will be small as both α_x and $1-\alpha_x$ are less than unity. The maximum value of this frequency occurs when $\alpha_x = (q-1)/(n-1)$, and it is round this value that the frequency clusters and clusters very narrowly if n be considerable. Hence we need a value of x, which if it increases in inexactness near the tails of the sampled population, shall be correct to a high degree of accuracy within the range where the frequency is sensible.

To attain this result the following steps were taken: (*a*) If α_x' be the area from the sampled population's mean to the ordinate at x, α_x' was expanded in terms of x.

On p. 369 we have $$\alpha_x' = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx,$$

or, expanding and integrating,

$$\alpha_x' = z_0 x \left(1 - \tfrac{1}{6}x^2 + \tfrac{1}{40}x^4 - \tfrac{1}{336}x^6 + \tfrac{1}{3456}x^8 + (-1)^s \frac{1.3.\ldots.(2s-1)}{(2s+1)!}x^{2s} + \ldots\right) \quad \ldots\ldots(\text{viii}),$$

where $z_0 = \dfrac{1}{\sqrt{2\pi}}$ = central ordinate of normal curve. If we only proceed as far as x^8 the value of α_x' is correct at the quartile to a unit in $2\frac{1}{2}$ million.

(*b*) We write $\alpha_x'/z_0 = \lambda$, and proceed to invert the above series. After somewhat lengthy algebra we find

$$x = \lambda\,(1 + \tfrac{1}{6}\lambda^2 + \tfrac{7}{120}\lambda^4 + \tfrac{127}{5040}\lambda^6 + \tfrac{4369}{362880}\lambda^8 + \tfrac{136}{24192}\lambda^{10} + \tfrac{7529}{1153152}\lambda^{12}) \quad \text{(ix)},$$

where the last term has been adjusted to allow for the remainder of the series.

(*c*) Now $\lambda = \alpha_x'/z_0 = (\alpha_x - 0{\cdot}5)/z_0$, hence if we substitute, i.e. use Taylor's Theorem, we can obtain an expansion of x in terms of α_x. The differential coefficients of x with regard to λ have to be taken to a large number of decimal places as they have afterwards to be multiplied by the powers of $\dfrac{1}{z_0}$, which are considerable.

The following expression was obtained:

$$\begin{aligned} x = &- 2{\cdot}160,944,236,002 + 5{\cdot}666,941,673,295\,\frac{\alpha_x}{z_0} \\ &- 13{\cdot}789,173,963,335\left(\frac{\alpha_x}{z_0}\right)^2 + 30{\cdot}226,546,535,503\left(\frac{\alpha_x}{z_0}\right)^3 \\ &- 50{\cdot}695,978,635,644\left(\frac{\alpha_x}{z_0}\right)^4 + 65{\cdot}042,157,128,601\left(\frac{\alpha_x}{z_0}\right)^5 \\ &- 64{\cdot}017,842,888,135\left(\frac{\alpha_x}{z_0}\right)^6 + 48{\cdot}337,003,737,513\left(\frac{\alpha_x}{z_0}\right)^7 \\ &- 27{\cdot}799,606,500,570\left(\frac{\alpha_x}{z_0}\right)^8 + 11{\cdot}976,952,491,413\left(\frac{\alpha_x}{z_0}\right)^9 \\ &- 3{\cdot}747,413,066,285\left(\frac{\alpha_x}{z_0}\right)^{10} + {\cdot}805,121,376,710\left(\frac{\alpha_x}{z_0}\right)^{11} \\ &- {\cdot}106,378,541,439\left(\frac{\alpha_x}{z_0}\right)^{12} + {\cdot}006,529,061,217\left(\frac{\alpha_x}{z_0}\right)^{13} \end{aligned} \quad \ldots\ldots(\text{x}).$$

(*d*) The value of the powers of $\dfrac{1}{z_0} = \sqrt{2\pi}$ were then found to twelve decimal places and multiplied into the corresponding coefficients, and thus the expression for x in terms of α_x was obtained. The values of the coefficients were computed to twelve decimals, but they are here cut down to nine. We have

$$\begin{aligned} x = &- 2{\cdot}160,944,236 + 14{\cdot}204,916,796\,\alpha_x - 86{\cdot}639,935,245\,\alpha_x^2 \\ &+ 476{\cdot}056,317,940\,\alpha_x^3 - 2001{\cdot}397,015,439\,\alpha_x^4 + 6436{\cdot}423,430,546\,\alpha_x^5 \\ &- 15879{\cdot}639,592,524\,\alpha_x^6 + 30054{\cdot}483,275,604\,\alpha_x^7 - 43326{\cdot}950,404,854\,\alpha_x^8 \\ &+ 46790{\cdot}289,738,035\,\alpha_x^9 - 36697{\cdot}029,289,753\,\alpha_x^{10} + 19762{\cdot}898,205,065\,\alpha_x^{11} \\ &- 6545{\cdot}355,530,736\,\alpha_x^{12} + 1006{\cdot}977,773,964\,\alpha_x^{13} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{xi}). \end{aligned}$$

This is our fundamental formula, and we shall write it in the form

$$x = -b_0 + b_1\alpha_x - b_2\alpha_x^2 + b_3\alpha_x^3 - b_4\alpha_x^4 + b_5\alpha_x^5 - b_6\alpha_x^6 + b_7\alpha_x^7 - b_8\alpha_x^8 + b_9\alpha_x^9 - b_{10}\alpha_x^{10} + b_{11}\alpha_x^{11} - b_{12}\alpha_x^{12} + b_{13}\alpha_x^{13} \quad \text{(xii)},$$

where the b's have known numerical values.

Substituting in Equation (i) and expressing the result in complete B-functions, we have, after reinstating σ, the population standard deviation:

$$\frac{{}_n\bar{x}_q}{\sigma} = \frac{\Gamma(n+1)}{\Gamma(q)\,\Gamma(n-q+1)}\{-b_0\mathrm{B}(q, n-q+1) + b_1\mathrm{B}(q+1, n-q+1) - \ldots + b_{12}\mathrm{B}(q+13, n-q+1)\}.$$

Whence, by replacing the B- by Γ-functions and the latter by factorials, we find

$$\frac{{}_n\bar{x}_q}{\sigma} = -b_0 + b_1\frac{q}{n+1} - b_2\frac{q(q+1)}{(n+1)(n+2)} + b_3\frac{q(q+1)(q+2)}{(n+1)(n+2)(n+3)} - b_4\frac{q(q+1)(q+2)(q+3)}{(n+1)(n+2)(n+3)(n+4)} + \ldots + b_{12}\frac{q(q+1)\ldots(q+12)}{(n+1)(n+2)\ldots(n+13)} \quad \text{(xiii)}.$$

This is the mean deviate of the individual in the qth rank in samples of n, drawn from an indefinitely large population of standard deviation σ, normally distributed.

The accuracy of this formula depends on two factors: (a) the accuracy with which (xii) gives x, and (b) the degree of concentration of the distribution of the rank about its modal value. Now we need only deal with ranks in which q is $> \frac{1}{2}n$, and (xii) will give results when x is not too great with about 5 or 6 decimal place accuracy. Thus when

Area from Tail	Formula (xii)	Actual Value	
$\alpha =$	$x =$	$x =$	Δ
·25	−·674,4897	−·674,4898	−·000,0001
·37	−·331,8531	−·331,8533	−·000,0002
·50	·000,0003	·000,0000	−·000,0003
·75	·674,4903	·674,4898	−·000,0005
·81	·877,9033	·877,8963	−·000,0070

(iv) *Mean Value of the Quartile for Samples of various sizes.*

Equation (xiii) was applied* to determine the mean value of the Quartile Rank. Consider first the two cases $n = 4m+3$ and $n = 4m+2$, in both of which the quartile, $3m+3$ and $3m+2$ respectively, is occupied by an individual. These require only a single use of (xiii), and there is no further choice in determining ${}_n\bar{x}_q$. If secondly we take the cases $n = 4m$ and $n = 4m+1$, there is no individual in the quartile, and it is often assumed that the best value for the quartile is the half of the adjacent individuals' deviations, i.e. one-half the values for $3m$ and $3m+1$ in the first case, and one-half the values for $3m+1$ and $3m+2$ in the second case. This halving of the sum of the two values appears to have been adopted without much consideration of whether it produces the best results.

* The arithmetical work was very laborious, but it was to some extent simplified by working out one series, and then deducing some of the others from it, by multiplying and dividing the several terms by appropriate factors.

TABLE I.

Mean Values of the Quartile Rank in terms of the Population Standard Deviation.

Value of n	$n=4m$	$n=4m+1$	$n=4m+2$	$n=4m+3$
1	·000,000			
	H. ·000,000			
2			·544,076	
			H. ·564,190	
3				·816,114
				H. ·846,284
4	·642,547			
	H. ·663,194			
5		·804,183		
		H. ·828,991		
6			·637,881	
			H. ·641,755	
7				·752,119
				H. ·757,375
8	·657,006			
	H. ·662,521			
9		·747,509		
		H. ·752,135		
10			·654,797	
			H. ·656,053	
11				·727,184
				H. ·728,847
12	·663,655+			
	H. ·664,627			
13		·724,902		
		H. ·726,366		
14			·661,224	
15				·714,184
16	·666,072			
17		·712,779		
18			·664,523	
19				·706,268
20	·667,580			
21		·705,337		
22			·666,514	
23				·700,960
24	·668,602			
25		·700,283		
30			·668,793	
31				·694,309
32	·669,961			
33		·693,928		
50			·671,165−	
51				·686,644
52	·671,601			
53		·686,495+		
70			·672,141	
71				·683,250
72	·672,367			
73		·683,171		
98			·672,825−	
99				·680,786
100	·672,942			
101		·680,745+		
∞	·674,490	·674,490	·674,490	·674,490

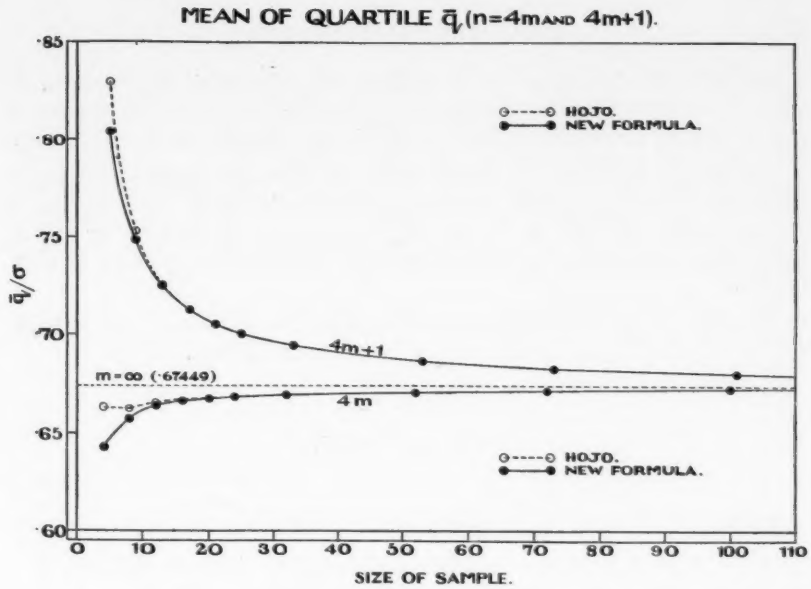

MEAN OF QUARTILE $\bar{q}$ (n=4m AND 4m+1).
·85
·80
·75
·70
·65
·60
$\bar{q}/\sigma$
HOJO.
NEW FORMULA.
4m+1
m=∞ (·67449)
4m
HOJO.
NEW FORMULA.
0 10 20 30 40 50 60 70 80 90 100 110
SIZE OF SAMPLE.

Fig. 1.

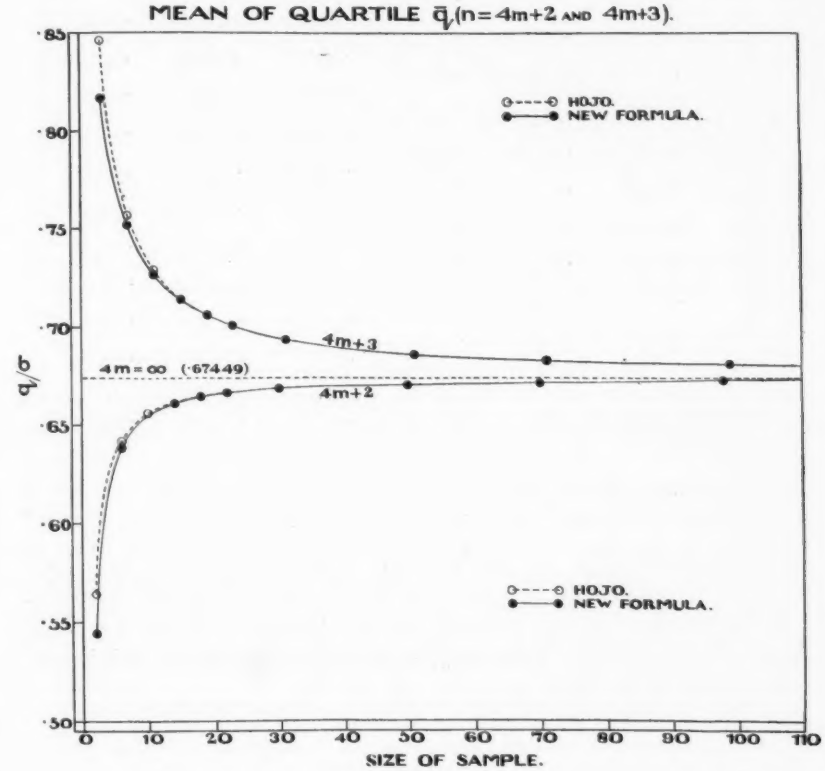

MEAN OF QUARTILE $\bar{q}$ (n=4m+2 AND 4m+3).
·85
·80
·75
·70
·65
·60
·55
·50
q/σ
HOJO.
NEW FORMULA.
4m+3
4m=∞ (·67449)
4m+2
HOJO.
NEW FORMULA.
0 10 20 30 40 50 60 70 80 90 100 110
SIZE OF SAMPLE.

Fig. 2.

In Table I the halving process is used for comparison with Hojo's results. It will be seen that our results for practical purposes coincide with Hojo's for samples from 10 to 20. They fill in the gap between his small samples and the large ones.

Professor Hojo's values, where they exist for the small samples at the top of this table, are to be preferred. They are given preceded by the letter H. But it is clear, that by 15 to 16 for all practical purposes they will run into our values. The results obtained for the quartile mean are represented graphically in Figs. 1 and 2.

We will now examine a little more at length the first two columns of our table which are obtained by taking the mean value of two individuals.

TABLE II.

($n = 4m$ *and* $4m+1$.)

Size of Sample n	$n=4m$			$n=4m+1$		
	$\xi_1 = {}_n\bar{x}_{3m}$	$\xi_2 = {}_n\bar{x}_{3m+1}$	$\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2$	$\xi_1 = {}_n\bar{x}_{3m+1}$	$\xi_2 = {}_n\bar{x}_{3m+2}$	$\frac{3}{4}\xi_1 + \frac{1}{4}\xi_2$
4	·295,413	·989,681	·642,547			
5				·492,355−	1·116,012	·648,269
8	·472,174	·841,839	·657,006			
9				·571,044	·923,974	·659,276
12	·536,579	·790,732	·663,655			
13				·602,486	·847,319	·663,694
16	·569,849	·762,296	·666,072			
17				·619,247	·806,311	·666,013
20	·590,204	·744,956	·667,580			
21				·629,701	·780,972	·667,519
24	·603,918	·733,286	·668,602			
25				·636,818	·763,750−	·668,551
32	·621,264	·718,658	·669,961			
33				·645,923	·741,933	·669,926
52	·641,515	·701,687	·671,601			
53				·656,675	·716,316	·671,585+
72	·650,603	·694,132	·672,367			
73				·661,546	·704,796	·672,359
100	·657,253	·688,631	·672,942			
101				·665,129	·696,361	·672,937
∞	·674,490	·674,490	·674,490	·674,490	·674,490	·674,490

As our Figs. 1 and 2 and Table I indicate, two of our series, those for $n = 4m+1$ and $n = 4m+3$ are descending towards the limit, and two, those for $n = 4m$ and $n = 4m+2$ are ascending towards it. The members of each pair of series are in good accordance with each other, but not with members of the other pair of series. Those for $n = 4m+1$ and $n = 4m+3$ deviate even at 100 very considerably from the limiting value. Table II indicates that in the case of $4m+1$ by taking $\frac{3}{4}{}_n\bar{x}_{3m+1} + \frac{1}{4}{}_n\bar{x}_{3m+2}$ instead of $\frac{1}{2}{}_n\bar{x}_{3m+1} + \frac{1}{2}{}_n\bar{x}_{3m+2}$ as is usually done, we can bring the

$n = 4m + 1$ set of values into very close accord with the sets for $n = 4m$, and $n = 4m + 2$. This process changes it into an ascending series as close to the limiting value in the neighbourhood of 100 as the other two series. The factors $\frac{3}{4}$, $\frac{1}{4}$ applied to the $n = 4m$ set of values, on the contrary, would destroy the pre-existing harmony based on factors $\frac{1}{2}, \frac{1}{2}$. It remains only to note that the set for $n = 4m + 3$ is still out of keeping with the other three. The quartile rank is here taken to be $3m + 3$. Having doubts as to the accuracy of this I calculated the mean values of the $3m + 2$ rank, and took $\frac{1}{4}{}_n\bar{x}_{3m+2} + \frac{3}{4}{}_n\bar{x}_{3m+3}$, with the result given in the following table.

TABLE III.

($n = 4m + 3$.)

	$\xi_1 = {}_n\bar{x}_{3m+2}$	$\xi_2 = {}_n\bar{x}_{3m+3}$	$\frac{1}{4}\xi_1 + \frac{3}{4}\xi_2$
3	·000,000	·816,113	·612,085⁻
7	·352,292	·752,119	·652,162
11	·461,794	·727,184	·660,837
15	·515,583	·714,184	·664,534
19	·547,637	·706,268	·666,610
23	·568,919	·700,960	·667,949
31	·595,433	·694,309	·669,590
51	·625,917	·686,644	·671,462
71	·639,432	·683,250⁻	·672,295⁺
99	·649,260	·680,786	·672,905ᵉ
∞	·674,490	·674,490	·674,490

It is now clear that we have all our four series in reasonable agreement, and all *ascend* to the limiting value. Table IV gives the combined results and Fig. 3 shows the four series running, for practical purposes, into one and the same curve at between 20 and 30. This seems a more satisfactory result than the column of Means in Professor Hojo's Table IV[a]*. At the same time it is merely a matter of definition as to what we wish to term the "quartile" rank. The "quartile" rank is merely a means of deducing from ranking the standard deviation of the distribution, and *once the values* of a series of ${}_n\bar{x}_q$'s are determined, there is no reason whatever for determining σ from the interquartile distance; as I have shown elsewhere it is at any rate for large samples rather a poor choice of ranks from which to deduce σ†.

Table III shows, however, that if we aim at getting σ from the quartiles, with the more or less customary definitions, we shall get exaggeratedly small values in the cases of $n = 4m$ and $4m + 2$, if we use the divider ·674,490, and exaggeratedly large values in the like way for the cases of $n = 4m + 1$ and $4m + 3$. Further, with such definitions, only two of the series even by $n = 100$ will approximate to the

* *Biometrika*, Vol. XXIII. p. 341.

† *Biometrika*, Vol. XIII. pp. 119—121.

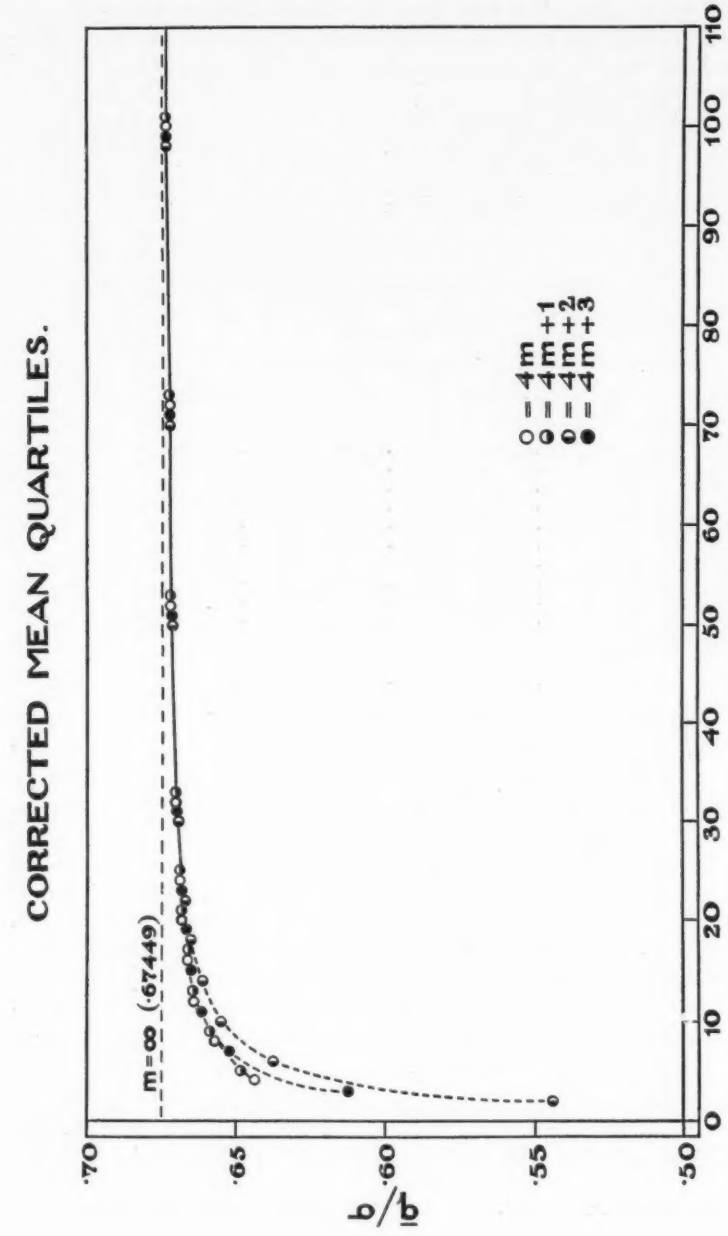

CORRECTED MEAN QUARTILES.
$\bar{q}/q$
·70
·65
·60
·55
·50
m=∞ (·67449)
○ = 4m
◐ = 4m+1
◑ = 4m+2
● = 4m+3
0
10
20
30
40
50
60
70
80
90
100
110
SIZE OF SAMPLE.

Fig. 8.

TABLE IV.

Size of Sample n	Mean "Quartile"	Size of Sample n	Mean "Quartile"
1	·000,000	22	·666,514
2	·544,076	23	·667,949
3	·612,085⁻	24	·668,602
4	·642,547	25	·668,551
5	·648,269	30	·668,793
6	·637,881	31	·669,590
7	·652,162	32	·669,961
8	·657,006	33	·669,926
9	·659,276	50	·671,165
10	·654,797	51	·671,462
11	·660,837	52	·671,601
12	·663,655⁺	53	·671,585
13	·663,694	70	·672,141
14	·661,224	71	·672,295⁺
15	·664,534	72	·672,367
16	·666,072	73	·672,359
17	·666,013	98	·672,825
18	·664,523	99	·672,905[e]
19	·666,610	100	·672,942
20	·667,580	101	·672,937
21	·667,519		
		∞	·674,490

limit ·674,490. On the other hand, Table IV shows us that with the following definitions of the Quartile:

$n = 4m$, Quartile $= \frac{1}{2}(3m)$th rank $+ \frac{1}{2}(3m+1)$th rank,

$n = 4m+1$, Quartile $= \frac{3}{4}(3m+1)$th rank $+ \frac{1}{4}(3m+2)$th rank,

$n = 4m+2$, Quartile $= (3m+1)$th rank,

$n = 4m+3$, Quartile $= \frac{1}{4}(3m+2)$th rank $+ \frac{3}{4}(3m+3)$th rank,

our means run with a considerable degree of smoothness and approach reasonably the limit for very large samples.

It is clear that in the case of $n = 4m$ and $n = 4m+2$ the dichotomic lines through $q = \frac{1}{2}(x_{3m} + x_{3m+1})$ and $q = 3m+2$ do cut off one-quarter of the individuals. But in the case of $n = 4m+1$ the dichotomic line $q = \frac{1}{2}(x_{3m+1} + x_{3m+2})$ cuts off only m and not $m + 0{\cdot}25$ individuals, and in the case of $n = 4m+3$ the dichotomic line at $q = 3m+3$ cuts off only $m + 0{\cdot}5$ and not $m + 0{\cdot}75$ individuals, the true quarter. Hence, in the first case, we need to emphasise the x_{3m+1} component in relation to the x_{3m+2}, and in the second case we need to introduce a portion of the preceding rank, namely x_{3m+2}.

Consider first the case of $n = 4m+1$, the deviate at the $(3m+1)$th individual may be supposed to bisect him, or his deviation x_{3m+1} corresponds to an $\alpha_{x_{3m+1}} = 3m + {\cdot}5$; in the same manner x_{3m+2} corresponds to $\alpha_{x_{3m+2}} = 3m + 1{\cdot}5$. But we need a value of $\alpha_{x_{3m+{\cdot}75}} = 3m + {\cdot}75$ at the quartile, or $\alpha_{3m+{\cdot}75} = \frac{1}{4}(3\alpha_{3m+5} + \alpha_{3m+1{\cdot}5})$,

and accordingly by linear interpolation $\bar{x}_{3m+\cdot75} = \frac{1}{4}(3\bar{x}_{3m+\cdot5} + \bar{x}_{3m+1\cdot5})$, or in our notation $= \frac{1}{4}(3\bar{x}_{3m+1} + \bar{x}_{3m+2})$.

Now consider the second case. To reach the true quartile we require to obtain the x corresponding to $\alpha_{3m+2\cdot25} = \frac{1}{4}(\alpha_{3m+1\cdot5} + 3\alpha_{3m+2\cdot5})$; but these latter two α's correspond to the dichotomic lines at the $(3m+2)$th and $(3m+3)$th individuals, or again assuming a linear interpolation the deviation at the quartile will be given by $\frac{1}{4}(\bar{x}_{3m+2} + 3\bar{x}_{3m+3})$, and this more closely than by the deviation of the individual in the $(3m+3)$th rank. Actually, as I have indicated, there is no special need to seek for the quartiles; any mirror-pair, one in the neighbourhood of the first and the other in the neighbourhood of the third quartile, will determine the mean more accurately than the median; and, at any rate for largish samples, a mirror-pair taken at about $\frac{1}{14}$th of the total number of individuals from either end of the ranking will give a far better result for the standard deviation than the quartiles, or a mirror-pair in their neighbourhood*. However, if we seek the quartiles, it is, I think, better to use the definitions given above, rather than those more usual, which place the quartile at an individual, if there be one in the corresponding rank, or take half of the characters of the adjacent ranks if their be no individual at the quartile.

(v) *On the Standard Deviation of any Rank in a Sample of Size n.*

Starting from equation (ix), squaring the result and proceeding as far as λ^{16}, we have

$$x^2 = \lambda^2 + \cdot333,333,333,333\lambda^4 + \cdot144,444,444,444\lambda^6 + \cdot069,841,269,841\lambda^8 + \cdot035,881,834,215^+\lambda^{10} + \cdot017,287,073,780\lambda^{12} + \cdot016,820,057,377\lambda^{14} + \cdot003,385,937,431\lambda^{16} \quad \text{.....(xiv).}$$

We now replace λ by α'_x/z_0, and substituting the powers of $1/z_0$ find:

$$x^2 = 6\cdot283,185,307\alpha'^2_x + 13\cdot159,472,535^-\alpha'^4_x + 35\cdot829,475,275^-\alpha'^6_x + 108\cdot850,793,790\alpha'^8_x + 351\cdot377,523,072\alpha'^{10}_x + 1063\cdot654,778,920\alpha'^{12}_x + 6502\cdot592,689,482\alpha'^{14}_x + 8224\cdot658,517,124\alpha'^{16}_x \quad \text{.....(xv).}$$

The accuracy of this formula was tested at $\alpha'_x = \cdot25$ and $\alpha'_x = \cdot30$.

At the former value $x^2 = \cdot454,936,271$, which differs in excess from the true value by $\cdot000,000,043$. At the latter value $x^2 = \cdot708,326,334$, which is in excess of the true value by $\cdot000,000,033$. Such accuracy is sufficient for our purpose.

We now note that $\alpha'_x = \alpha - 0\cdot5$ and therefore

$$\alpha'^2_x = \tfrac{1}{4} - \alpha_x(1-\alpha_x) = \tfrac{1}{4} - \gamma, \text{ say.}$$

We therefore insert this value of α'^2_x in equation (xv) and obtain a series in γ:

$$x^2 = 4\cdot503,505,870 - 54\cdot608,093,113\alpha_x(1-\alpha_x) + 387\cdot654,090,067\alpha_x^2(1-\alpha_x)^2 - 2035\cdot495,696,318\alpha_x^3(1-\alpha_x)^3 + 7350\cdot284,493,202\alpha_x^4(1-\alpha_x)^4 - 17678\cdot088,798,881\alpha_x^5(1-\alpha_x)^5 + 26836\cdot344,390,480\alpha_x^6(1-\alpha_x)^6 - 22951\cdot909,723,730\alpha_x^7(1-\alpha_x)^7 + 8224\cdot658,517,124\alpha_x^8(1-\alpha_x)^8 \quad \text{.....(xvi).}$$

* See *Biometrika*, Vol. XIII. pp. 119 and 121.

Testing this equation on $a_x = \cdot 75$, or $a_x(1-a_x) = \cdot 1875$, we find $x^2 = \cdot 454{,}936{,}271$, absolutely identical with the value obtained above, or the somewhat troublesome transformation has been correctly made. We will write Equation (xvi) in the form

$$x^2 = c_0 - c_1 a_x(1-a_x) + c_2 a_x^2(1-a_x)^2 - c_3 a_x^3(1-a_x)^3 + c_4 a_x^4(1-a_x)^4 - c_5 a_x^5(1-a_x)^5 + c_6 a_x^6(1-a_x)^6 - c_7 a_x^7(1-a_x)^7 + c_8 a_x^8(1-a_x)^8 \quad \ldots\ldots(\text{xvi})^{bis},$$

and then can express the standard deviation ${}_n\sigma_q$ of the qth rank in a sample of size n by the equation

$$\begin{aligned} {}_n\sigma_q^2 + {}_n\bar{x}_q^2 &= \frac{\Gamma(n+1)}{\Gamma(q)\,\Gamma(n-q+1)} \int_0^1 a_x^{q-1}(1-a_x)^{n-q}\, x^2 da_x \\ &= \frac{\Gamma(n+1)}{\Gamma(q)\,\Gamma(n-q+1)} [c_0 \mathrm{B}(q, n-q+1) - c_1 \mathrm{B}(q+1, n-q+2) \\ &\quad + c_2 \mathrm{B}(q+2, n-q+3) + c_3 \mathrm{B}(q+3, n-q+4) \\ &\quad + c_4 \mathrm{B}(q+4, n-q+5) - c_5 \mathrm{B}(q+5, n-q+6) \\ &\quad + c_6 \mathrm{B}(q+6, n-q+7) + c_7 \mathrm{B}(q+7, n-q+8) \\ &\quad + c_8 \mathrm{B}(q+8, n-q+9)]. \end{aligned}$$

Hence expressing the B-functions in terms of Γ-functions, and cancelling factors, we have

$$\begin{aligned} {}_n\sigma_q^2 = c_0 &- \frac{q(n-q+1)}{(n+1)(n+2)} c_1 + \frac{q(q+1)(n-q+1)(n-q+2)}{(n+1)(n+2)(n+3)(n+4)} c_2 \\ &- \frac{q(q+1)(q+2)(n-q+1)(n-q+2)(n-q+3)}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} c_3 \\ &+ \ldots + \frac{q(q+1)\ldots(q+7)(n-q+1)(n-q+2)\ldots(n-q+8)}{(n+1)(n+2)\ldots(n+8)(n+9)(n+10)\ldots(n+16)} c_8 - {}_n\bar{x}_q^2 \end{aligned} \quad \ldots\ldots(\text{xvii}).$$

We are now in a position to find the standard deviation of any rank q in a sample of any size n from an indefinitely large normal population. Rather to illustrate the method than to give a complete tabular view of ${}_n\sigma_q$ for samples of different sizes and individuals of different ranks, I give on p. 380 a table of the standard deviation of the quartile for samples of $n = 4m+2$. In this case the true quartile does dichotomise an individual rank, and the deviation of that rank ($q = 3m+2$) corresponds to the accurate quartile.

I do not think it needful to deal with the standard deviations of the so-called "quartiles" when they do not coincide with an exact rank. Any pair of mirror-ranks on either side the quartiles will give a good value for the median, and the quartiles are not the best ranks from which to determine the standard deviation of the sampled population.

Figs. 4 and 5 show the data of Table I in graphic form. Upon them I have placed Professor Hojo's values. His values are to be preferred to mine for the smaller samples. But mine run into his very rapidly, say, for practical purposes,

TABLE V.

Standard Deviation of the Quartile σ_q, where $n = 4m + 2$.

Size of Sample	Ratio of Quartile Standard Deviation σ_q to Population σ	Ratio of Quartile Standard Deviation σ_q to $\sigma/\sqrt{n}$, Standard Deviation of Mean
2	·777,371	1·099,367
6	·516,739	1·265,749
10	·414,016	1·308,920
14	·354,623	1·326,878
18	·314,940	1·336,177
22	·286,064	1·341,914
30	·246,121	1·348,060
50	·191,535	1·354,357
70	·162,178	1·356,879
98	·137,233	1·358,537
∞	·000,000	1·362,632

about samples of 15 to 20. From that size onwards they bridge the gulf till at about samples of 100 we may safely use the asymptotic value recorded at the foot

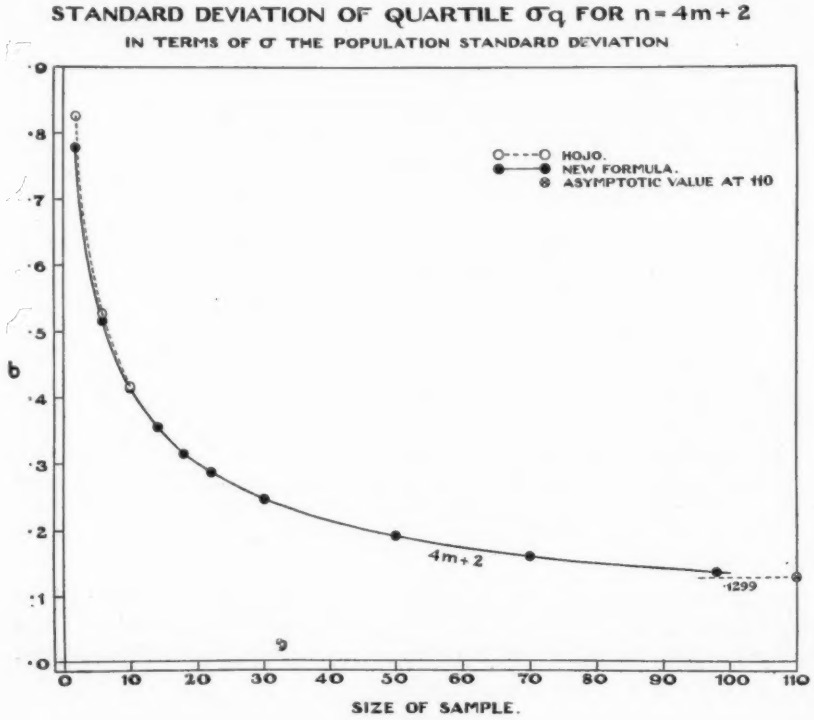


Fig. 4.

of the last column. Thus if we needed to find the standard error in the quartile rank in a sample of 150, we should take

$$\sigma_q = 1{\cdot}362{,}632\,\frac{\sigma}{\sqrt{n}} = \frac{1{\cdot}362{,}632}{12{\cdot}247{,}4487}\,\sigma = {\cdot}111{,}258\sigma.$$

The difference at 100 is less than 0·4 %.

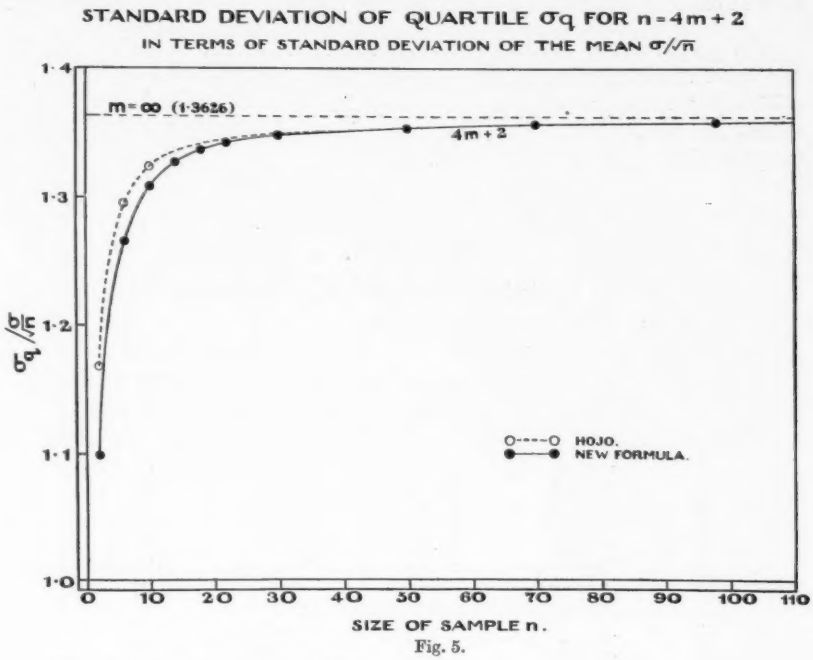


Fig. 5.

(vi) *Standard Deviation of the Median.*

We now turn to the standard deviation of the median, and shall confine our attention to the case, when it is occupied by an individual, i.e. $n = 2m + 1$.

Turning to equation (xvii), we have ${}_n\bar{v}_q = 0$, $q = m + 1$ and $n = 2m + 1$. Hence the standard deviation of the median, $\sigma_{\text{med.}}$, is given by

$$\begin{aligned}
\sigma_{\text{med.}}^2 = c_0 &- \tfrac{1}{2}\frac{m+1}{2m+3}c_1 + \tfrac{1}{4}\frac{(m+1)(m+2)}{(2m+3)(2m+5)}c_2 - \tfrac{1}{8}\frac{(m+1)(m+2)(m+3)}{(2m+3)(2m+5)(2m+7)}c_3 \\
&+ \tfrac{1}{16}\frac{(m+1)(m+2)(m+3)(m+4)}{(2m+3)(2m+5)(2m+7)(2m+9)}c_4 \\
&- \tfrac{1}{32}\frac{(m+1)(m+2)(m+3)(m+4)(m+5)}{(2m+3)(2m+5)(2m+7)(2m+9)(2m+11)}c_5 \\
&+ \tfrac{1}{64}\frac{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}{(2m+3)(2m+5)(2m+7)(2m+9)(2m+11)(2m+13)}c_6 \\
&- \tfrac{1}{128}\frac{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)(m+7)}{(2m+3)(2m+5)(2m+7)(2m+9)(2m+11)(2m+13)(2m+15)}c_7 \\
&+ \tfrac{1}{256}\frac{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)(m+7)(m+8)}{(2m+3)(2m+5)(2m+7)(2m+9)(2m+11)(2m+13)(2m+15)(2m+17)}c_8
\end{aligned}$$

.....(xviii).

Substituting for m successively the values 0, 1, 2, etc. we obtain the following table for the ratios $\sigma_{\text{med.}}/\sigma$ and $\sigma_{\text{med.}}/(\sigma/\sqrt{n})$. The corresponding values as found by Hojo are also tabled. His values are preferable when we have to deal with samples of 10 or less, but for practical purposes mine are in agreement with his from 11 onwards, and bridge the gulf from his values up to the asymptotic values.

The general run of the results is shown in Figs. 6 and 7. It is clear that the new formula while agreeing to four places of figures with Hojo's in the neighbourhood of 15, has not reached accordance with the asymptotic value in the fourth place even at 101. But for the great arithmetical labour involved, it would have been well to compute values in the second hundred.

TABLE VI.

Standard Deviation of the Median $\sigma_{\text{med.}}$ *for* $n = 2m + 1$.

	$\sigma_{\text{med.}}/\sigma$		$\sigma_{\text{med.}}/(\sigma/\sqrt{n})$	
n	New Formula	Hojo	New Formula	Hojo
1	·948,854	1·00000	·948,854	1·00000
3	·644,251	·66983	1·150,516	1·16018
5	·534,527	·53557	1·195,238	1·19757
7	·458,485+	·45874	1·213,038	1·21371
9	·407,477	·40755	1·222,432	1·22265
11	·370,386	·37044	1·228,433	1·22861
13	·341,463	—	1·231,163	—
15	·318,692	—	1·234,288	—
17	·300,076	—	1·237,244	—
19	·284,237	—	1·238,959	—
21	·270,665	—	1·240,343	—
23	·258,867	—	1·241,483	—
25	·248,488	—	1·242,439	—
—	—	—	—	—
31	·223,530	—	1·244,561	—
33	·216,832	—	1·245,607	—
—	—	—	—	—
51	·174,757		1·248,011	—
53	·171,455+	Asymptotic values	1·248,212	—
—	—		—	—
71	·148,290	·148,741	1·249,511	—
73	·146,256	·146,689	1·249,615+	—
—	—	—	—	—
99	·125,689	·125,963	1·250,589[5]	—
101	·124,444	·124,709	1·250,642	—
∞	—	—	1·253,314	—

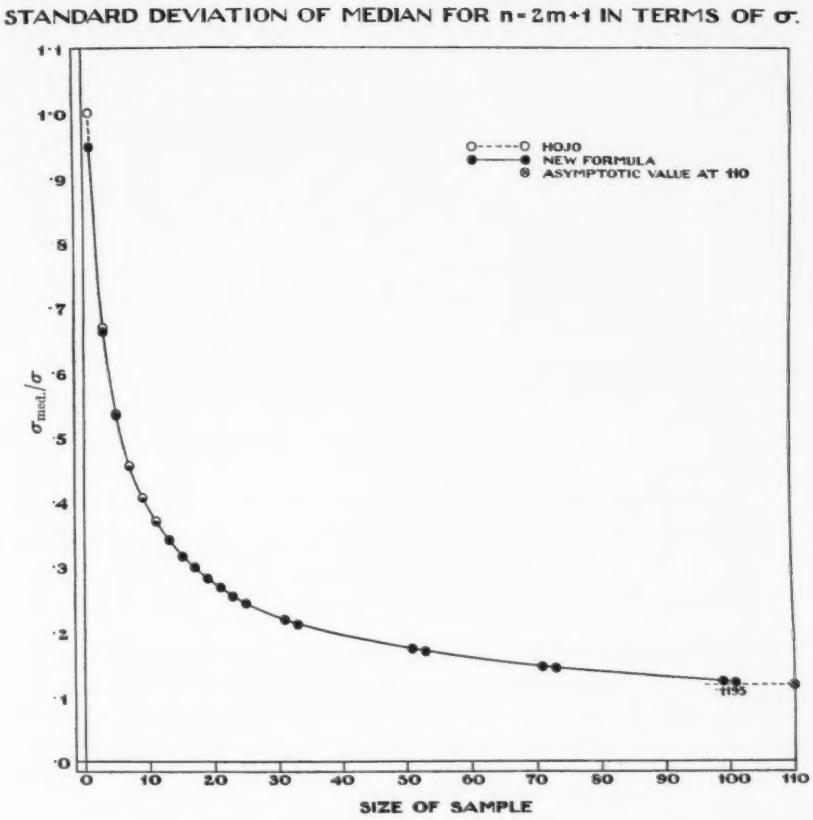

STANDARD DEVIATION OF MEDIAN FOR n=2m+1 IN TERMS OF σ.
HOJO
NEW FORMULA
ASYMPTOTIC VALUE AT 110
σmed./σ
1·1
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
·0
1155
0
10
20
30
40
50
60
70
80
90
100
110
SIZE OF SAMPLE

Fig. 6.

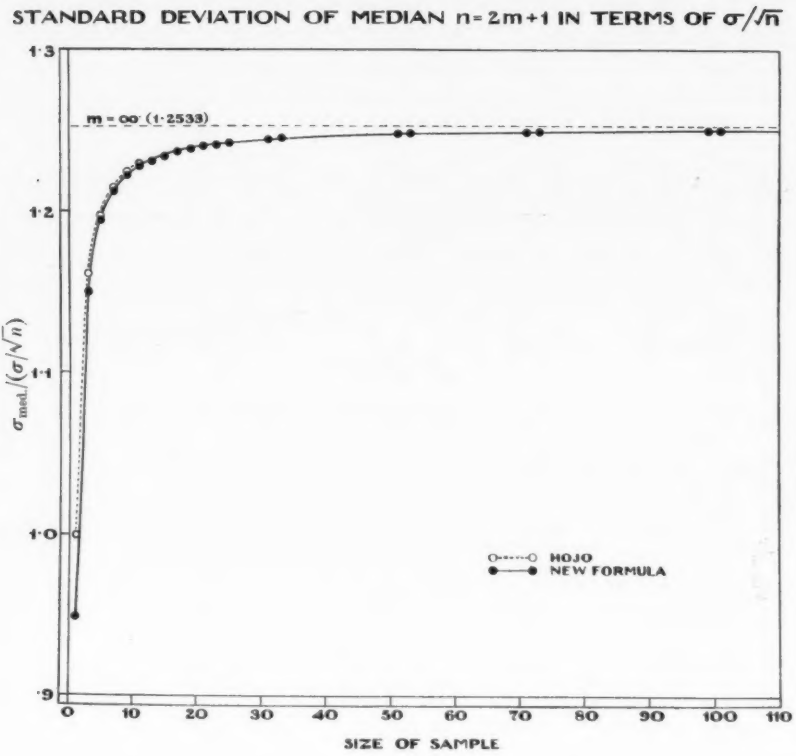


Fig. 7.

(vii) *Correlation of the Characters of Individuals in the qth and q'th Ranks.*

Let us now consider the mean product of the variates x_q and $x_{q'}$ corresponding to the qth and q'th ranks. Then representing mean $(x_q x_{q'})$ by $\{x_q x_{q'}\}$ we have

$$\{x_q x_{q'}\} = \int_0^1 \int_{a_{q'}}^1 (1-a_q)^{q-1}(a_q - a_{q'})^{q'-q-1} a_{q'}^{n-q'} x_q x_{q'} da_q da_{q'} \times \frac{n!}{(q-1)!\,(q'-q-1)!\,(n-q')!\,1!\,1!},$$

where if $f(x)$ be the law of frequency of the sampled population, with limits ξ_1 and ξ_2 of x,

$$a_x = \int_x^{\xi_2} f(x)\,dx, \quad 1-a_x = \int_{\xi_1}^{x} f(x)\,dx \quad \text{and} \quad \int_{\xi_1}^{\xi_2} f(x)\,dx = 1.$$

We suppose our ranking to proceed from right to left, so that q is less than q' but $a_q > a_{q'}$. Rank 1 has the largest variate.

Now let us suppose it possible to invert the integral for α_x and expand x in powers of α_x, thus:

$$x_q = \underset{s=0}{S}\,(-1)^{s+1} b_s \alpha^s{}_q, \qquad x_{q'} = \underset{t=0}{S}\,(-1)^{t+1} b'_t\, \alpha^t{}_{q'}.$$

Then

$$\{x_q x_{q'}\} = \underset{s}{S}\,\underset{t}{S}(-1)^{s+t+2} b_s b'_t \times \frac{n!}{(q-1)!\,(q'-q-1)!\,(n-q')!}$$
$$\times \int_0^1 \int_{\alpha_{q'}}^1 (1-\alpha_q)^{q-1} (\alpha_q - \alpha_{q'})^{q'-q-1} \alpha_{q'}{}^{n-q'+t}\, \alpha_q{}^s\, d\alpha_q d\alpha_{q'}.$$

Calling the double integral $I_{s,t}$ we will now proceed to its evaluation*.

Let
$$u = \int_{\alpha_{q'}}^1 (1-\alpha_q)^{q-1} (\alpha_q - \alpha_{q'})^{q'-q-1} \alpha_q{}^s\, d\alpha_q.$$

Then
$$I_{s,t} = \int_0^1 \alpha_{q'}{}^{n-q'+t}\, u\, d\alpha_{q'} = \int_0^1 \frac{u\, d\alpha_{q'}{}^{n-q'+t+1}}{n-q'+t+1}$$
$$= \left[\frac{u\alpha_{q'}{}^{n-q'+t+1}}{n-q'+t+1}\right]_0^1 - \frac{\int_0^1 \alpha_q{}^{n-q'+t+1} \dfrac{du}{d\alpha_{q'}}\, d\alpha_{q'}}{n-q'+t+1}.$$

Now $u=0$ when $\alpha_{q'}=1$ and $\alpha_{q'}=0$ at the lower limit; hence the first term between limits vanishes. We have next to find $du/d\alpha_{q'}$. This is given by

$$\frac{du}{d\alpha_{q'}} = \left[-(1-\alpha_q)^{q-1} (\alpha_q - \alpha_{q'})^{q'-q-1} \alpha_q{}^s\right]_{\alpha_q=\alpha'_q}$$
$$-(q'-q-1)\int_{\alpha_{q'}}^1 (1-\alpha_q)^{q-1} (\alpha_q - \alpha_{q'})^{q'-q-2} \alpha_q{}^s\, d\alpha_q.$$

The term between square brackets vanishes and we have

$$I_{s,t} = \frac{q'-q-1}{n-q'+t+1} \int_0^1 \alpha_{q'}{}^{n-q'+t+1} \int_{\alpha_{q'}}^1 (1-\alpha_q)^{q-1} (\alpha_q - \alpha_{q'})^{q'-q-2} \alpha_q{}^s d\alpha_q d\alpha_{q'}.$$

This is precisely the same result as we had before except that: (i) there is a multiplying factor $(q'-q-1)/(n-q'+t+1)$, (ii) the power of $\alpha_{q'}$ has been raised by unity and (iii) the power of $\alpha_q - \alpha_{q'}$ has been reduced by unity. If we repeat this process continually we shall get rid of the $\alpha_q - \alpha_{q'}$ term, and we reach ultimately

$$I_{s,t} = \frac{(q'-q-1)(q'-q-2)\ldots 1}{(n-q'+t+1)(n-q'+t+2)\ldots(n-q'+t+q'-q-1)}$$
$$\times \int_0^1 \alpha_{q'}{}^{n-q'+t+q'-q-1} \int_{\alpha_{q'}}^1 (1-\alpha_q)^{q-1} \alpha_q{}^s d\alpha_q d\alpha_{q'}.$$

We have now to find the integral on the right.

* Clearly this integral is entirely independent of any idea of normal distribution. The method here and elsewhere adopted in this paper applies to any case where we can expand x in powers of α_x. For example, in the case of a rectangular population $x_q = b_1 \alpha_{q'}$ results can be obtained at once. See the concluding section of this paper.

Let $v = \int_{a_{q'}}^{1} (1-a_q)^{q-1} a_q^{s} da_q$, then we need

$$I' = \int_0^1 a_{q'}^{n+t-q-1} v da_{q'}.$$

Integrating by parts,

$$I' = \left[\frac{v a_{q'}^{n+t-q}}{n+t-q}\right]_0^1 - \int_0^1 \frac{a_{q'}^{n+t-q}}{n+t-q} \frac{dv}{da_{q'}} da_{q'}.$$

The term between limits vanishes, and we have

$$I' = \frac{\int_0^1 a_{q'}^{n+t+s-q} (1-a_{q'})^{q-1} da_{q'}}{n+t-q}$$

$$= \frac{\mathrm{B}(n+t+s-q+1,\, q)}{n+t-q}$$

$$= \frac{\Gamma(n+t+s-q+1)\,\Gamma(q)}{(n+t-q)\,\Gamma(n+t+s+1)}$$

$$= \frac{(n+t+s-q)!\,(q-1)!}{(n+t-q)\,(n+t+s)!}.$$

Accordingly

$$I_{s,t} = \frac{(q'-q-1)!\,(n+t-q')!\,(q-1)!\,(n+t+s-q)!}{(n-q+t)!\,(n+t+s)!}.$$

And finally

$$\{x_q x_{q'}\} = \underset{s}{S}\,\underset{t}{S} (-1)^{s+t+2} b_s b'_t \frac{n!}{(n-q')!} \frac{(n+t-q')!\,(n+t+s-q)!}{(n-q+t)!\,(n+t+s)!} \qquad \text{(xix)}.$$

Hence if we can expand x_q in the powers of a_q in the case of any frequency distribution whatsoever, we can determine the mean interval between any two ranks for any sized sample, and the standard error of this interval; for

$$\left.\begin{aligned} \bar{\imath}_{q'-q} &= \bar{x}_{q'} - \bar{x}_q \\ \text{and} \qquad \sigma^2_{q'-q} &= \{x^2_{q'}\} + \{x^2_q\} - 2\{x_q x_{q'}\} - (\bar{x}_{q'} - \bar{x}_q)^2 \end{aligned}\right\} \quad \text{.....(xx)},$$

$\bar{x}_{q'}$ and $\bar{x}_q$ are given by Equation (xiii), p. 371, $\{x^2_{q'}\}$ and $\{x^2_q\}$ by Equation (xvii), p. 379 and twice $\{x_q x_{q'}\}$ by the result just written down. It is perhaps unnecessary to state that the correlation of x_q and $x_{q'}$ may be determined by the mean value $\{x_q x_{q'}\}$ and our knowledge of $\bar{x}_{q'}$, $\bar{x}_q$, $\sigma_{q'}$ and σ_q. Returning to Equation (xix), we note that b'_t cannot be put equal to b_t until we have evaluated the factorial terms. b'_t corresponds to $x_{q'}$ and to $a_{q'}$, and the expression for $\{x_q x_{q'}\}$—since q' is always greater than q—is not symmetrical in s and t.

We can now write down the lengthy formula introducing the necessary σ^2, the variance of the sampled population. We note on comparing the coefficient of b'_s as arranged in Equation (xxi) with Equation (xiii) that the part in curled brackets is with negative sign the mean value of the variate of the $(n-q+1+s)$th rank in samples of size $n+s$, or more precisely $= -{}_{n+s}\bar{x}_{n-q+1+s}/\sigma$.

For $q' > q$

$$\{x_q x_{q'}\}/\sigma^2 = b_0' \left\{b_0 - \frac{n-q+1}{n+1} b_1 + \frac{(n-q+1)(n-q+2)}{(n+1)(n+2)} b_2 - \ldots\right\}$$
$$- b_1' \left(\frac{n-q'+1}{n+1}\right) \left\{b_0 - \frac{n-q+2}{n+2} b_1 + \frac{(n-q+2)(n-q+3)}{(n+2)(n+3)} b_2 - \ldots\right\}$$
$$+ b_2' \frac{(n-q'+1)(n-q'+2)}{(n+1)(n+2)} \left\{b_0 - \frac{n-q+3}{n+3} b_1 + \frac{(n-q+3)(n-q+4)}{(n+3)(n+4)} b_3 - \ldots\right\}$$
$$+ b_3' \frac{(n-q'+1)(n-q'+2)(n-q'+3)}{(n+1)(n+2)(n+3)} \left\{b_0 - \frac{n-q+4}{n+4} b_1 + \frac{(n-q+4)(n-q+5)}{(n+4)(n+5)} b_2 - \ldots\right\}$$
$$+ \ldots\ldots\ldots\ldots$$
$$+ b_s' \frac{(n-q'+1)\ldots(n-q'+s)}{(n+1)\ldots(n+s)} \left\{b_0 - \frac{n-q+1+s}{n+1+s} b_1 + \frac{(n-q+1+s)(n-q+1+s+1)}{(n+1+s)(n+1+s+1)} b_2 - \ldots\right\}$$
$$+ \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\

Accordingly

$$\{x_q x_{q'}\}/\sigma^2 = b_0 {}_q\bar{x}_n - b_1 \frac{n-q'+1}{n+1} {}_q\bar{x}_{n+1} + b_2 \frac{(n-q'+1)(n-q'+2)}{(n+1)(n+2)} {}_q\bar{x}_{n+2} - b_3 \frac{(n-q'+1)(n-q'+2)(n-q'+3)}{(n+1)(n+2)(n+3)} {}_q\bar{x}_{n+3} + \ldots \quad \ldots(\text{xxi})^{ter}.$$

As a corollary we may consider the special case of q and q' being ranks at equal distances from the start and finish of the ranking. In this case $n-q' = q-1$ or $n-q'+1 = q$, and we have

$$\{x_q x_{q'}\}/\sigma^2 = b_0 {}_q\bar{x}_n - b_1 \frac{q}{n+1} {}_q\bar{x}_{n+1} + b_2 \frac{q(q+1)}{(n+1)(n+2)} {}_q\bar{x}_{n+2} - b_3 \frac{q(q+1)(q+2)}{(n+1)(n+2)(n+3)} {}_q\bar{x}_{n+3} + \ldots \quad \ldots\ldots\ldots\ldots\ldots(\text{xxii}).$$

We must not, of course, put $q = q'$ in the subscripts of x_q and $x_{q'}$ on the left of this equation.

We will illustrate this formula arithmetically by considering (i) the correlation between the two quartiles when $n = 10$ and 22, or $q = 3$ and 6 respectively, (ii) the mean interquartile lengths for these cases, and (iii) their variances for the normal population.

Case (i). $n = 10$, $q = 3$.

We shall need ${}_3\bar{x}_{10}, {}_3\bar{x}_{11}, {}_3\bar{x}_{12}, \ldots {}_3\bar{x}_{23}$. These must be found from Equation (xiii), where the b's are provided in Equation (xi). We have

${}_3\bar{x}_{10} = -\cdot654,796,106$	${}_3\bar{x}_{17} = -1\cdot024,249,979$
${}_3\bar{x}_{11} = -\cdot727,183,358$	${}_3\bar{x}_{18} = -1\cdot059,735,761$
${}_3\bar{x}_{12} = -\cdot790,730,987$	${}_3\bar{x}_{19} = -1\cdot092,807,604$
${}_3\bar{x}_{13} = -\cdot847,218,445$	${}_3\bar{x}_{20} = -1\cdot123,245,536$
${}_3\bar{x}_{14} = -\cdot897,944,456$	${}_3\bar{x}_{21} = -1\cdot151,844,968$
${}_3\bar{x}_{15} = -\cdot943,885,434$	${}_3\bar{x}_{22} = -1\cdot178,647,392$
${}_3\bar{x}_{16} = -\cdot985,790,013$	${}_3\bar{x}_{23} = -1\cdot203,839,003.$

In addition we know that ${}_q\bar{x}_{10} = -{}_{q'}\bar{x}_{10}$, and that $\sigma_{x_q} = \sigma_{x_{q'}} = \cdot413,917,27$. The corresponding multipliers are

$b_0 = 2\cdot160,944,236$	$-\frac{9}{4862}\ b_7 = -55\cdot633,556,043$
$-\frac{3}{11}\ b_1 = -3\cdot874,068,217$	$+\frac{5}{4862}\ b_8 = 44\cdot556,715,760$
$+\frac{1}{11}\ b_2 = 7\cdot876,357,749$	$-\frac{5}{8398}\ b_9 = -27\cdot857,995,795$
$-\frac{5}{143}\ b_3 = -16\cdot645,325,802$	$+\frac{3}{8398}\ b_{10} = 13\cdot109,293,138$
$+\frac{15}{1001}\ b_4 = 29\cdot990,964,267$	$-\frac{1}{4522}\ b_{11} = -4\cdot370,388,811$
$-\frac{1}{143}\ b_5 = -45\cdot009,954,060$	$+\frac{1}{7106}\ b_{12} = \cdot921,102,664$
$+\frac{1}{286}\ b_6 = 55\cdot523,215,358$	$-\frac{15}{163,438}\ b_{13} = -\cdot092,418,327$

Multiplying the former series by the latter we have

$$\{x_q x_{q'}\}/\sigma^2 = -152{\cdot}336{,}465{,}979 + 151{\cdot}974{,}990{,}392$$
$$= -{\cdot}361{,}475.$$
$$r_{qq'} = \frac{-{\cdot}361{,}475 - (-{\cdot}654{,}796{,}106 \times {\cdot}654{,}796{,}738)}{({\cdot}413{,}919{,}27)^2}$$
$$= {\cdot}392{,}717.$$

Hojo (p. 357) has ·3717, undoubtedly a better value, as my formula is not good enough at $n = 10$; the asymptotic value is ·3333.

Substituting in the formula

$$\sigma^2_{q'-q} = \sigma^2_q + \sigma^2_{q'} - 2\sigma_q \sigma_{q'} r_{qq'},$$

we find $\sigma_{q'-q} = {\cdot}46700 \times \sigma$. Here Hojo has the value $\cdot49757 \times \sigma$, again to be preferred*.

Case (ii). We may now turn to the case of $n = 22$ and shall find the correlation of the quartile and the standard error of the interquartile distance. Here $q = 6$, and we have

${}_6\bar{x}_{22} = -{\cdot}666{,}513{,}568$	${}_6\bar{x}_{29} = -{\cdot}870{,}270{,}060$
${}_6\bar{x}_{23} = -{\cdot}700{,}958{,}943$	${}_6\bar{x}_{30} = -{\cdot}893{,}731{,}400$
${}_6\bar{x}_{24} = -{\cdot}733{,}302{,}322$	${}_6\bar{x}_{31} = -{\cdot}916{,}127{,}587$
${}_6\bar{x}_{25} = -{\cdot}763{,}767{,}429$	${}_6\bar{x}_{32} = -{\cdot}937{,}547{,}973$
${}_6\bar{x}_{26} = -{\cdot}792{,}544{,}792$	${}_6\bar{x}_{33} = -{\cdot}958{,}066{,}583$
${}_6\bar{x}_{27} = -{\cdot}819{,}797{,}709$	${}_6\bar{x}_{34} = -{\cdot}977{,}749{,}397$
${}_6\bar{x}_{28} = -{\cdot}845{,}667{,}250$	${}_6\bar{x}_{35} = -{\cdot}996{,}655{,}530.$

Further we have ${}_{17}\bar{x}_{22} = -{}_6\bar{x}_{22}$† and

$$\sigma_q = \sigma_{q'} = {\cdot}286{,}063{,}995 \times \sigma.$$

The factors we need to multiply the above values of ${}_6x_{22+s}$ by are:

$+b_0 = +2{\cdot}160{,}944{,}236$	$-\frac{22}{43{,}355}b_7 = -15{\cdot}250{,}804{,}568$
$-\frac{6}{23}b_1 = -3{\cdot}705{,}630{,}468^5$	$+\frac{11}{50{,}025}b_8 = +9{\cdot}527{,}165{,}506$
$+\frac{7}{92}b_2 = +6{\cdot}592{,}168{,}986$	$+\frac{154}{155{,}0775}b_9 = -4{\cdot}646{,}518{,}431$
$-\frac{14}{575}b_3 = -11{\cdot}590{,}936{,}437$	$-\frac{77}{165{,}4160}b_{10} = +1{\cdot}708{,}221{,}245$
$+\frac{63}{7475}b_4 = +16{\cdot}867{,}961{,}468$	$-\frac{7}{310{,}155}b_{11} = -{\cdot}446{,}035{,}974$
$-\frac{14}{4485}b_5 = -20{\cdot}091{,}399{,}783$	$+\frac{7}{620{,}310}b_{12} = +{\cdot}073{,}862{,}244$
$+\frac{11}{8970}b_6 = +19{\cdot}473{,}359{,}589$	$-\frac{3}{516{,}925}b_{13} = -{\cdot}005{,}844{,}046.$

* The mean interquartile distance is $1{\cdot}309{,}593 \times \sigma$ as against Hojo's $1{\cdot}312{,}106 \times \sigma$.

† Actually from our approximate formula we found

$${}_{17}\bar{x}_{22} = {\cdot}666{,}514{,}193 \times \sigma \text{ and } {}_6\bar{x}_{22} = -{\cdot}666{,}513{,}568 \times \sigma.$$

Multiplying the former series of values by the latter we have

$$\{x_q x_{q'}\}/\sigma^2 = -46{\cdot}299{,}434{,}029 + 45{\cdot}883{,}528{,}543$$
$$= -{\cdot}415{,}905{,}486,$$
$$r_{qq'} = \frac{\{x_q x_{q'}\}/\sigma^2 - \{\bar{x}_q \bar{x}_{q'}\}/\sigma^2}{\sigma_q \sigma_{q'}/\sigma^2}$$
$$= \frac{-{\cdot}415{,}905{,}486 + {\cdot}444{,}240{,}753}{{\cdot}081{,}832{,}609}$$
$$= {\cdot}346{,}259.$$

Hence
$$\sigma^2_{q'-q} = 2\sigma^2_q (1 - r_{qq'})$$
$$= {\cdot}163{,}665{,}218 \times (1 - {\cdot}346{,}259)\, \sigma^2$$
$$= {\cdot}106{,}994{,}684,\ \sigma^2,$$

whence
$$\sigma_{q'-q} = {\cdot}327{,}100 \times \sigma.$$

If we had used the asymptotic value of $r_{qq'}$ at the quartiles, we should have found

$$\sigma_{q'-q} = {\cdot}330{,}318 \times \sigma.$$

Using the complete asymptotic expression for $\sigma_{q'-q}$*—i.e. $2{\cdot}2252\sigma/\sqrt{2n}$—we find

$$\sigma_{q'-q} = {\cdot}335{,}462 \times \sigma.$$

Thus clearly we obtain a better result, if the standard deviation of the quartile be known, in using that and the asymptotic value of the interquartile correlation to determine the standard deviation of the interquartile distance, than if we use the full asymptotic value. It seems highly probable that for many practical purposes in measuring the accuracy of interrank distances the above process would be adequate for samples of 30 or over. In this way the troublesome process of finding the interrank correlation would be avoided.

(viii) *Application of the General Formulae of this Paper to Ranking in Samples from a Rectangular Population*†.

Let us suppose the sampled population to be one for which all values of the character under investigation between a_1 and a_2 are equally likely to occur, and let $a_2 - a_1 = b$. Take a_1 as the origin of x; then if h be the height of the rectangle its total frequency $= hb$. We have $x = ba_x$, and while a_x ranges from 0 to 1, x ranges from 0 to b. If σ be the standard deviation of the parent population

$$\sigma^2 = b^2/12 \quad \text{and} \quad b = 2\sqrt{3}\,\sigma = 3{\cdot}464{,}1016\,\sigma,$$

we find at once

$${}_n\bar{x}_q = b\frac{n+1-q}{n+1} = 3{\cdot}464{,}1016\,\frac{n+1-q}{n+1}\,\sigma \quad \text{.....(xxiii),}$$

$$\sigma_q^2 = \frac{b^2 q(n-q+1)}{(n+1)^2(n+2)} = \frac{12q(n-q+1)}{(n+1)^2(n+2)}\,\sigma^2 \text{.....(xxiv),}$$

$$\{x_q x_{q'}\} = \frac{b^2(n+1-q')(n+2-q)}{(n+1)(n+2)} \quad \text{.....(xxv),}$$

* *Biometrika*, Vol. XIII. p. 119, Table II, Column 3.

† Many of the results of this section have been obtained previously by J. Neyman and E. S. Pearson. See *Biometrika*, Vol. XX^A, p. 208 *et seq*. The methods of approach are, however, different.

$$p_{11}=\{x_q x_{q'}\}-{}_n\bar{x}_q\times{}_n\bar{x}_{q'}=b^2\left[\frac{(n+1-q')(n+2-q)}{(n+1)(n+2)}-\frac{(n+1-q')(n+1-q)}{(n+1)^2}\right]$$

$$=\frac{b^2q(n+1-q')}{(n+1)^2(n+2)}\quad\text{.....(xxvi),}$$

$$r_{qq'}=p_{11}/\sigma_q\sigma_{q'}=\sqrt{\frac{q(n+1-q')}{q'(n+1-q)}}\quad\text{.....(xxvii).}$$

If q and q' be mirror-ranks, or $q-1=n-q'$, we have

$$r_{qq'}=q/q'\quad\text{.....(xxvii)}^{bis}.$$

In general

$$\sigma^2_{q'-q}=\sigma^2_{q'}+\sigma^2_q-2p_{11}$$

$$=\frac{b^2q(n+1-q)+q'(n+1-q')-2q(n+1-q')}{(n+1)^2(n+2)}$$

$$=\frac{b^2(q'-q)(n+1-q'+q)}{(n+1)^2(n+2)}\quad\text{.....(xxviii).}$$

If q' and q again be mirror-ranks, or $q-1=n-q'$, then we find

$$\text{Mean Range}=b\left(\frac{n+1-2q}{n+1}\right)\quad\text{.....(xxix)}$$

and the standard deviation is given by

$$\sigma^2_{\text{range}}=\frac{2b^2q(n+1-2q)}{(n+1)^2(n+2)}\quad\text{.....(xxx).}$$

The successive moments of the distribution of the distance from origin to the qth rank are easily seen to be

$$\left.\begin{aligned}&\mu_1'=b\frac{q}{n+1}, && \mu_2'=b^2\frac{q(q+1)}{(n+1)(n+2)}\\ &\mu_3'=b^3\frac{q(q+1)(q+2)}{n(n+1)(n+2)}, && \mu_4'=b^4\frac{q(q+1)(q+2)(q+3)}{(n+1)(n+2)(n+3)(n+4)}\end{aligned}\right\}\text{....(xxxi),}$$

and so on. Accordingly* the curve of distribution of the character in the qth rank in N samples is

$$y=\frac{N}{b}\frac{(n-1)!}{(q-1)!(n-q)!}\left(\frac{q-1}{n-1}+\frac{x}{b}\right)^{q-1}\left(\frac{n-q}{n-1}-\frac{x}{b}\right)^{n-q}\quad\text{.....(xxxii),}$$

the origin being at the mode $\frac{q-1}{n-1}b$. This is a Pearson curve of Type I†. The distribution cannot become even approximately normal unless *both* $q-1$ and $n-q$ are large, i.e. for the distribution of rank characters near the mode. The distribution of the character of the first individual ($q=1$) is given by

$$y=\frac{N}{b}\left(1-\frac{x}{b}\right)^{n-1}\quad\text{.....(xxxiii),}$$

* See *Phil. Trans.*, Vol. 186A, p. 368.
† See *Phil. Trans.*, Vol. 216A, p. 441.

a curve of Pearson's Type IX*, which starts with a finite ordinate at $x=0$. The distribution of character in the median rank, if $n=2p+1$, is given by the symmetrical curve of Pearson Type II†,

$$y=\frac{N}{b}\frac{(2p)!}{p!\,p!}\left(\frac{1}{4}-\frac{x^2}{b^2}\right)^p \quad \text{.....(xxxiv).}$$

This approaches a normal curve when n is very large.

Again, if we desire the sth moment of the range between the qth and q'th ranks, we have

$$\mu_s'=\frac{n!}{(q-1)!(q'-q-1)!\,(n-q')!}\int_0^1 d\alpha_x{}^{q-1}\int_{\alpha_q}^1(\alpha_{x'}-\alpha_x)^{q'-q-1}(1-\alpha_{x'})^{n-q'}\alpha_x{}^{q-1}\times(x_{q'}-x_q)^s d\alpha_{x'}.$$

But $$(x_{q'}-x_q)^s=(\alpha_{x'}-\alpha_x)^s\times b^s,$$

so that $$\mu'_s=C_{q,q'}b^s\int_0^1\alpha_x{}^{q-1}u_s d\alpha_x,$$

where $$C_{q,q'}=\frac{n!}{(q-1)!\,(q'-q-1)!\,(n-q)!},$$

and $$u_s=\int_{\alpha_q}^1(\alpha_{x'}-\alpha_x)^{q'-q-1+s}(1-\alpha_{x'})^{n-q'}d\alpha_{x'}.$$

Now if we integrate by parts with regard to α_x we find as we did before that

$$\mu'_s=C_{qq'}b^s\frac{q'-q-1+s}{q}\int_0^1\alpha_x{}^q u_{s-1}d\alpha_x.$$

Proceeding in this way we ultimately exhaust the power of $\alpha_{x'}-\alpha_x$, raising the power of α_x by a unit each time. Thus we reach

$$\mu'_s=C_{qq'}b^s\frac{(q'-q-1+s)!\,(q-1)!}{(q'+s-2)!}\int_0^1\alpha_x{}^{q'+s-2}\int_{\alpha_q}^1(1-\alpha_{x'})^{n-q'}d\alpha_{x'}d\alpha_x$$

$$=C_{qq'}b^s\frac{(q'-q-1+s)!\,(q-1)!}{(q'+s-2)!}\frac{1}{n-q'+1}\int_0^1\alpha_x{}^{q'+s-2}(1-\alpha_x)^{n-q'+1}d\alpha_x.$$

Substituting for the complete B-function and for $C_{qq'}$ we find

$$\mu'_s=b^s\frac{n!\,(q'-q-1+s)!}{(n+s)!\,(q'-q-1)!} \quad \text{.....(xxxv).}$$

Thus we have

$$\mu_0'=1,\quad \mu_1'=b\frac{q'-q}{n+1},\quad \mu_2'=b^2\frac{(q'-q)(q'-q+1)}{(n+1)(n+2)},$$

leading to $$\sigma^2{}_{q'-q}=b^2\frac{(q'-q)(n+1-q'+q)}{(n+1)(n+2)},$$

and giving the mean range and standard error of that range between any two ranks, $q<q'$, $q'>q$.

* Because every moment-coefficient is the same for this curve and the frequency distribution, and by the nature of the case the curve is non-sinuous.

† *Phil. Trans.*, Vol. 186A, p. 372.

Proceeding in this way we find

$$\mu_3' = \frac{b^3(q'-q)(q'-q+1)(q'-q+2)}{(n+1)(n+2)(n+3)},$$

$$\mu_4' = \frac{b^4(q'-q)(q'-q+1)(q'-q+2)(q'-q+3)}{(n+1)(n+2)(n+3)(n+4)},$$

and so on to any value of s.

But these are the moments about one end of the range of the Pearson Type I curve*

$$y = \frac{N}{b}\frac{n!}{(q'-q-1)!\,(n-q'+q)!}\left(\frac{q'-q-1}{n-1}+\frac{w}{b}\right)^{q'-q-1}\left(\frac{n-q'+q}{n-1}-\frac{w}{b}\right)^{n-q'+q} \quad \text{.....(xxxvi),}$$

the origin being the mode and the range b. This curve therefore† gives the distribution of the intervals w between individuals in the qth and q'th ranks when we take N samples of size n from a rectangular population.

If $n = 4p+3$ and we take the quartiles as $q = p+1$, $q' = 3p+3$, we have

$$q'-q-1 = 2p+1 = n-q'+q = 2p+1 = \tfrac{1}{2}(n-1),$$

and accordingly our curve becomes

$$y = \frac{N}{b}\frac{(4p+3)!}{(2p+1)!\,(2p+1)!}\left(\frac{1}{4}-\left(\frac{w}{b}\right)^2\right)^{2p+1} \quad \text{.....(xxxvii),}$$

which is symmetrical, about the half range.

If, however, we take $n = 4p+2$, $q = p+1$, $q' = 3p+2$, we get the asymmetrical curve

$$y = \frac{N}{b}\frac{(4p+2)!}{(2p+1)!\,(2p)!}\left(\frac{2p}{4p+1}+\frac{x}{b}\right)^{2p}\left(\frac{2p+1}{4p+1}-\frac{x}{b}\right)^{2p+1} \quad \text{...(xxxviii).}$$

If we take the interval between the qth and q'th ranks, or $q' = q+1$, we find for the distribution curve

$$y = \frac{N}{b}n\left(1-\frac{w}{b}\right)^{n-1} \quad \text{.....(xxxix),}$$

or the distribution of intervals is independent of the value of q, being the same for all pairs of ranks. This is a curve of Pearson's Type IX, starting with a finite ordinate and finite slope when $w = 0$, and touching the axis with high contact at $w = b$. The mean interval is $b/(n+1)$, which for n considerable is close to $x = 0$. The standard deviation is $b\sqrt{n/(n+1)(n+2)}$, but this means little owing to the skewness of the J-curve giving the distribution.

If we put $q = 1$ and $q' = n$, we shall get the distribution of range in sampling from a rectangular population. This is a curious Type I curve‡, namely

$$y = \frac{N}{b}n(n-1)\left(\frac{n-2}{n-1}+\frac{w}{b}\right)^{n-2}\left(\frac{1}{n-1}-\frac{w}{b}\right) \quad \text{.....(xl).}$$

* See *Phil. Trans.* Vol. 186A (1895), p. 368.

† By the nature of the case a sinuous frequency distribution is excluded.

‡ See J. Neyman and E. S. Pearson, *Biometrika*, xxA, p. 210.

The mean is at $b(n-1)/(n+1)$, the mode at $b(n-2)/(n-1)$ and the standard deviation $= b\sqrt{2(n-1)/(n+1)(n+2)}$. The curve is extremely skew even for n moderately large, e.g. $=30$ say. It has high contact when $w=-\dfrac{n-2}{n-1}b$, i.e. for vanishing range and falls with almost vertical abruptness at the maximum range b.

It is clear that for most of the illustrations we have been considering normal distributions are far from being even approximations. It may we think be legitimately asked: Which Charlier series would reproduce the variety of distributions here reached? If it be replied: "But here the Pearson curves are theoretically indicated," then we may ask: What theoretical investigation has ever led to a Charlier distribution? And the answer must be for obvious reasons: None ever can.

Lastly we may deal with a case which concerns the "centre" of an interval. Such a centre has been defined by Neyman and E. S. Pearson as $\frac{1}{2}(\bar{x}_q+\bar{x}_{q'})$. The sth moment coefficient of such interval centre will accordingly be given by

$$\mu_s'=\frac{b^s n!}{2^s(q-1)!(q'-q-1)!(n-q')!}\int_0^1 a_x^{q-1}da_x\int_{a_x}^1(a_{x'}-a_x)^{q'-q-1}(1-a_{x'})^{n-q'}\times(a_{x'}+a_x)^s\,da_{x'}\quad\ldots\ldots(\text{xli}),$$

where the moment-coefficient is taken about the start of the rectangular population *.

For the particular case where we are dealing with the centre of the range, we have $q=1$, $q'=n$, and

$$\mu_s'=\frac{b^s n!}{2^s(n-2)!}\int_0^1 da_x\int_{a_x}^1(a_{x'}-a_x)^{n-2}(a_{x'}+a_x)^s\,da_{x'}\quad\ldots\ldots\ldots(\text{xli})^{bis}.$$

To evaluate this moment-coefficient we put

$$(a_{x'}+a_x)^s=\sum_{t=0}^{s}\frac{s!}{(s-t)!\,t!}a_{x'}^{s-t}a_x^t,$$

and write

$$u_{n-2}=\int_{a_x}^1(a_{x'}-a_x)^{n-2}a_{x'}^{s-t}\,da_{x'}.$$

Accordingly

$$\mu_s'=(\tfrac{1}{2}b)^s n(n-1)\int_0^1 a_x^t u_{n-2}da_x\times\frac{s!}{(s-t)!\,t!},$$

or integrating by parts

$$\int_0^1 a_x^t u_{n-2}da_x=\left[\frac{a_x^{t+1}}{t+1}u\right]_0^1-\int_0^1\frac{a_x^{t+1}}{t+1}\frac{du_{n-2}}{da_x}da_x.$$

But

$$\frac{du_{n-2}}{da_x}=\left[(a_{x'}-a_x)^{n-2}a_{x'}^{s-t}\right]_{a_x}-(n-2)\int_{a_x}^1(a_{x'}-a_x)^{n-3}a_{x'}^{s-t}da_{x'}$$
$$=-(n-2)\,u_{n-3},$$

or

$$\int_0^1 a_x^t u_{n-2}da_x=\frac{n-2}{t+1}\int_0^1 a_x^{t+1}u_{n-3}da_x.$$

* Equation (xli) can be applied to determine the "centre" and standard deviation of other intervals than the total range. For example, of a quartile or any decile, when there is no individual occupying that position, and we define the quartile or decile as the half of the characters of the adjacent ranks.

This process can be repeated until we exhaust the powers of $(a_{x'}-a_x)$ and reach $u_0=1$; in doing this we raise the power of a_x to $t+n-2$. Thus we obtain

$$\mu_s' = (\tfrac{1}{2}b)^s \frac{n(n-1)(n-2)\dots 1}{(t+1)(t+2)\dots t+n-2}\int_0^1 a_x{}^{t+n-2}\int_{a_x}^1 a_{x'}{}^{s-t}\,da_{x'}\,da_x \times \frac{s!}{(s-t)!\,t!}.$$

But
$$\int_{a_x}^1 a_{x'}{}^{s-t}\,da_{x'} = \frac{1-a_x{}^{s-t+1}}{s+t+t},$$

and
$$\int_0^1 \frac{a_x{}^{t+n-2}(1-a_x{}^{s-t+1})}{s-t+1}\,da_x = \frac{1}{s+t+1}\left[\frac{a_x{}^{t+n-1}}{t+n-1} - \frac{a_x{}^{n+s}}{n+s}\right]_0^1$$
$$= \frac{1}{(n+t-1)(n+s)}.$$

Thus
$$\mu_s' = (\tfrac{1}{2}b)^s \sum_{t=0}^{s} \frac{s!\,n!}{(n+s)(s-t)!\,(n+t-1)!}.$$

From this we deduce:

$$\mu_0 = 1,\quad \mu_1' = \tfrac{1}{2}b,\quad \mu_2' = \tfrac{1}{4}b^2\left(1+\frac{2}{(n+1)(n+2)}\right),$$

$$\mu_3' = \tfrac{1}{8}b^3\left(1+\frac{6}{(n+1)(n+3)}+\frac{6}{(n+1)(n+2)(n+3)}\right),$$

$$\mu_4' = \tfrac{1}{16}b^4\left(1+\frac{12}{(n+1)(n+4)}+\frac{24}{(n+1)(n+2)(n+3)(n+4)}\right),$$

$$\mu_5' = \tfrac{1}{32}b^5\left(1+\frac{20}{(n+1)(n+2)}+\frac{120}{(n+1)(n+2)(n+3)(n+4)}\right),$$

$$\mu_6' = \tfrac{1}{64}b^6\left(1+\frac{30}{(n+1)(n+6)}+\frac{120}{(n+1)(n+2)(n+3)}+\frac{720}{(n+1)(n+2)(n+3)(n+4)(n+5)}\right),$$

and thence by transferring to the mean we have

$$\mu_2 = (\tfrac{1}{2}b)^2\frac{2}{(n+1)(n+2)},\quad \text{or}\quad \sigma = \tfrac{1}{2}b\sqrt{\frac{2}{(n+1)(n+2)}},$$

$$\mu_3 = 0 = \mu_5 = \mu_{2i+1},$$

$$\mu_4 = (\tfrac{1}{2}b)^4\frac{4!}{(n+1)(n+2)(n+3)(n+4)},$$

$$\mu_6 = (\tfrac{1}{2}b)^6\frac{6!}{(n+1)(n+2)(n+3)\dots(n+6)},$$

and generally
$$\mu_{2i} = (\tfrac{1}{2}b)^{2i}\frac{(2i)!}{(n+1)(n+2)(n+3)\dots(n+2i)} \qquad \text{.....(xlii).}$$

We may now transform this result

$$\mu_{2i} = (\tfrac{1}{2}b)^{2i}\frac{n(n-1)!\,(2i)!}{(n+2i)!}$$
$$= (\tfrac{1}{2}b)^{2i}\,n\,\frac{\Gamma(n)\,\Gamma(2i+1)}{\Gamma(n+2i+1)} = (\tfrac{1}{2}b)^{2i}\,nB(n,\,2i+1)$$
$$= (\tfrac{1}{2}b)^{2i}\,n\int_0^1 z^{2i}(1-z)^{n-1}\,dz.$$

Put $z = \dfrac{2x}{b}$ and

$$\mu_{2i} = (\tfrac{1}{2}b)^{2i}\, n \int_0^{\frac{1}{2}b} \left(\frac{2}{b}\right)^{2i} x^{2i} \left(1 - \frac{2x}{b}\right)^{n-1} \frac{2}{b}\, dx$$

$$= \frac{2n}{b} \int_0^{\frac{1}{2}b} x^{2i} \left(1 - \frac{2x}{b}\right)^{n-1} dx.$$

Noting the 2 we see that μ_{2i} is the $2i$th moment-coefficient of the curve

$$y = \frac{Nn}{b}\left(1 - \frac{2x}{b}\right)^{n-1},$$

and its mirror-curve about the axis $x = 0$. Since from symmetry all the odd moments will vanish, all the moments of the distribution we are seeking accord with those of the double curve above, and since the required distribution between the limits $x = \pm \frac{1}{2}b$ is continuous and non-sinuous, the curve thus reached must be that of the distribution of the centre. It is the curve reached by Neyman and E. S. Pearson by a different and more direct method of approach*.

Summary. This paper is to be looked upon as an Appendix to the paper by Professor Hojo. Approximate formulae have been obtained for finding the mean value of any rank in samples of size n taken from a normal population. The formulae, however, apply to any case in which it is possible to express the abscissa of a frequency distribution in powers of the total frequency up to the ordinate corresponding to that abscissa. Unfortunately in the case of a normal distribution the series for x in terms of α_x is such that many terms have to be included, and it is only then semi-convergent. For very small samples and for very extreme ranks, it is liable to give sensible deviation from the correct values. Still we can deduce values which link up closely with Hojo's at about $n = 15$, and fill the gap between his and the asymptotic values. We are thus able to give a general solution of Galton's Problem†, namely that of finding the mean interval between any two individuals in a ranked series, and of finding the standard error of that interval. The discovery of the correlation between two ranks is the most laborious part of the method, but it seems highly probable that with samples of 30 and over, it would be adequate for practical purposes to use the asymptotic value of the correlation coefficient.

The ordinary definition of the quartile when it does not fall on a rank, i.e. to take it as half the sum of the characters in the adjacent ranks, is shown to be unsatisfactory, and a method of improving it is provided. At the same time there is no special advantage in the use of the quartile, and the discovery of its standard error where it does not fall on a rank is very laborious. It is part of the old and rather vicious practice of using the "probable error," which has given to the quartile a prominence in ranking which it does not deserve. Any pair of mirror-ranks in its neighbourhood will give practically as good a value for the median, and we can go a good way out from the quartiles and continue to better our

* *Biometrika*, xx[A], p. 210.

† See *Biometrika*, Vol. I. p. 385.

estimate of the standard deviation. The memoir also enables us to determine when the mean is known, the error we are not likely to exceed in supposing the character of the individual in the sth rank to be that of the corresponding individual in the sampled population.

In conclusion, apart from my principal collaborator, I have to thank very heartily my colleagues Miss E. M. Elderton, Miss B. Stoessiger and Mr E. C. Fieller for aid in the arithmetic involved in this paper. The computing work has been more laborious than may, perhaps, appear on the surface, and when the original computers were growing somewhat weary our colleagues came to our aid in completing the final series of values, and in checking a portion of our earlier work.

Part II of this paper to appear in the next issue applies the general formulae here obtained to certain skew distributions.

THE LANARKSHIRE MILK EXPERIMENT.

By "STUDENT."

In the spring of 1930* a nutritional experiment on a very large scale was carried out in the schools of Lanarkshire.

For four months 10,000 school children received $\frac{3}{4}$ pint of milk per day, 5000 of these got raw milk and 5,000 pasteurised milk, in both cases Grade A (Tuberculin tested); another 10,000 children were selected as controls and the whole 20,000 children were weighed and their height was measured at the beginning and end of the experiment.

It need hardly be said that to carry out an experiment of this magnitude successfully requires organisation of no mean order and the whole business of distribution of milk and of measurement of growth reflects great credit on all those concerned.

It may therefore seem ungracious to be wise after the event and to suggest that had the arrangement of the experiment been slightly different the results would have carried greater weight, but what follows is written not so much in criticism of what was done in 1930 as in the hope that in any further work full advantage may be taken of the light which may be thrown on the best methods of arrangement by the defects as well as by the merits of the Lanarkshire experiment.

The 20,000 children were chosen in 67 schools, not more than 400 nor less than 200 being chosen in any one school, and of these half were assigned as "feeders" and half as "controls," some schools were provided with raw milk and the others with pasteurised milk, no school getting both.

This was probably necessary for administrative reasons, owing to the difficulty of being sure that each of as many as 200 children gets the right kind of milk every day if there were a possibility of their getting either of the two. Nevertheless, as I shall point out later, this does introduce the possibility that the raw and pasteurised milks were tested on groups of children which were not strictly comparable.

Secondly, the selection of the children was left to the Head Teacher of the school and was made on the principle that both "controls" and "feeders" should be representative of the average children between 5 and 12 years of age: the actual method of selection being important I quote from Drs Leighton and McKinlay's* Report: "The teachers selected the two classes of pupils, those getting milk and those acting as "controls," in two different ways. In certain cases they selected them by ballot and in others on an alphabetical system." So far so good, but after invoking

* Department of Health for Scotland. *Milk Consumption and the Growth of Schoolchildren.* By Dr Gerald Leighton and Dr Peter L. McKinlay. (Edinburgh and London: H.M. Stationery Office, 1930.)

the goddess of chance they unfortunately wavered in their adherence to her for we read: "In any particular school where there was any group to which these methods had given an undue proportion of well fed or ill nourished children, others were substituted in order to obtain a more level selection." This is just the sort of afterthought that most of us have now and again and which is apt to spoil the best laid plans. In this case it was a fatal mistake, for in consequence the controls were, as pointed out in the Report*, definitely superior both in weight and height to the "feeders" by an amount equivalent to about 3 months' growth in weight and 4 months' growth in height.

Presumably this discrimination in height and weight was not made deliberately, but it would seem probable that the teachers, swayed by the very human feeling that the poorer children needed the milk more than the comparatively well to do, must have unconsciously made too large a substitution of the ill-nourished among the "feeders" and too few among the "controls" and that this unconscious selection affected, secondarily, both measurements.

Thirdly, it was clearly impossible to weigh such large numbers of children without impedimenta. They were weighed in their indoor clothes, with certain obvious precautions, and the difference in weight between their February garb and their somewhat lighter clothing in June is thus necessarily subtracted from their actual increase in weight between the beginning and end of the experiment. Had the selection of "controls" and "feeders" been a random one, this fact, as pointed out in the Report*, would have mattered little, both classes would have been affected equally, but since the selection was probably affected by poverty it is reasonable to suppose that the "feeders" would lose less weight from this case than the "controls." It is therefore not surprising to find that the gain in weight of "feeders" over "controls," which includes this constant error, was more marked, relatively to their growth rate, than was their gain in height, which was fortunately not similarly affected.

Fourthly, the "controls" from those schools which took raw milk were bulked with those from the schools which took pasteurised milk.

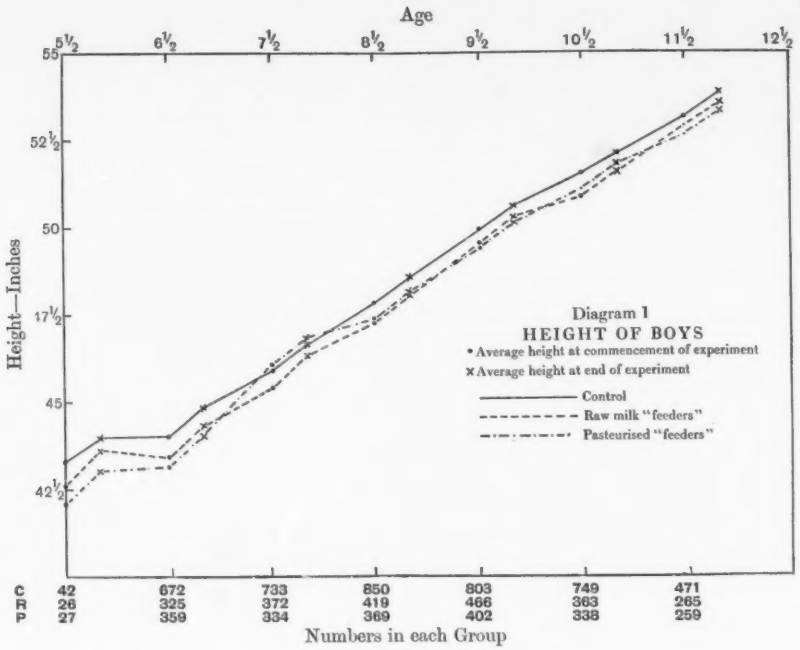
Now with only 67 schools, at best 33 against 34, in a district so heterogeneous both racially and socially, it is quite possible that there was a difference between the averages of the pupils at 33 schools and those of the pupils at another 34 schools both in the original measurements and in the rate of growth during the experiment.

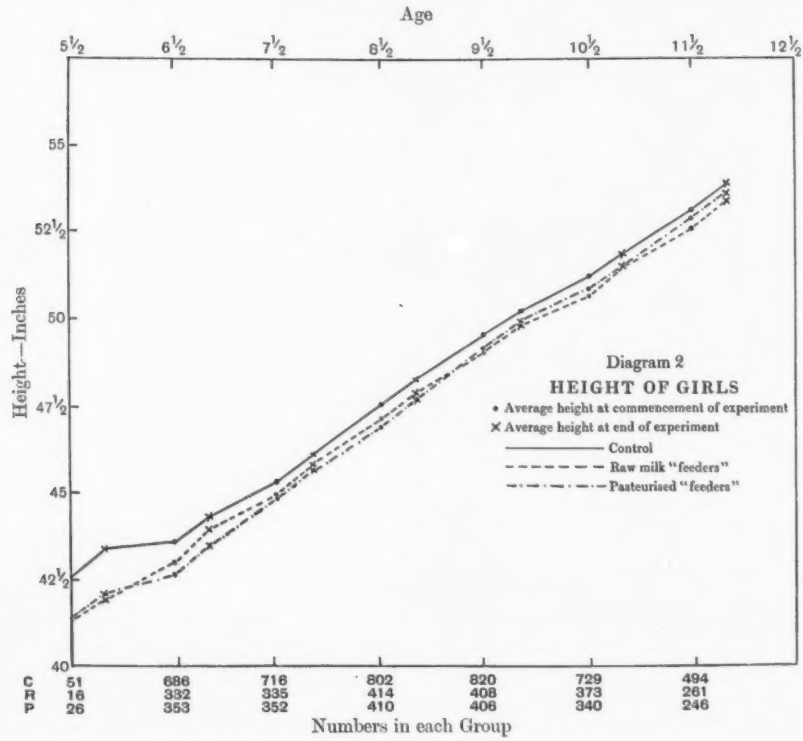
In that case the average "control" could not be used appropriately to compare with either the "raw" group or the "pasteurised" group.

This possibility is enhanced by the aforementioned selection of "controls" which can hardly have been carried out in a uniform manner in different schools.

Fortunately it would still be possible to correct this, for the figures for the different schools must still be available in the archives.

* See footnote on p. 398.


Age
5½ 6½ 7½ 8½ 9½ 10½ 11½ 12½
55
52½
50
47½
45
42½
Height—Inches
Diagram 1
HEIGHT OF BOYS
• Average height at commencement of experiment
× Average height at end of experiment
Control
Raw milk "feeders"
Pasteurised "feeders"
C 42 672 733 850 803 749 471
R 26 325 372 419 466 363 265
P 27 359 334 369 402 338 259
Numbers in each Group


Age
5½ 6½ 7½ 8½ 9½ 10½ 11½ 12½
55
52½
50
47½
45
42½
40
Height—Inches
Diagram 2
HEIGHT OF GIRLS
• Average height at commencement of experiment
× Average height at end of experiment
Control
Raw milk "feeders"
Pasteurised "feeders"
C 51 686 716 802 820 729 494
R 16 332 335 414 408 373 261
P 26 353 352 410 406 340 246
Numbers in each Group

Diagrams 1 and 2 give the average heights of "controls," raw milk "feeders" and pasteurised milk "feeders" for boys and girls respectively. The heights at the beginning of the experiments are set out against a uniform age scale centring each group at the half year above the whole number. This is doubtless accurate enough except for the first group aged "5 and less than 6," which was very much smaller in numbers than the other groups, either because only the older (or larger) children are sent to school between 5 and 6 or because the teachers did not think that the smaller children would be able to play their part. For this reason they should probably be centred more to the right compared to the others. A similar argument might lead us to centre the "11 and over" group a little more to the left.

The average heights at the end of the experiment are of course set out four months to the right of those at the beginning and it will be noticed that except for the first group, which is clearly out of place, not any of the points diverge very much from their appropriate line of growth whether "controls," "raws" or "pasteurised."

The case is very different in Diagrams 3 and 4 which show the corresponding average weights. Here there is, after the first two ages, a very decided dip, especially in the later ages. The weights at the end of the experiment are too low. This might be accounted for by a tendency in older children to grow normally in height and subnormally in weight during the spring, but I think it much more likely that older children wear about 1 lb. more clothes in February than they do in June, while in the case of younger children a more limited wardrobe permits of fewer discards.

The authors have tried to show that the selection of the "controls" has not affected the validity of the comparison, by computing the correlation coefficients between the original heights (and weights) and the growth during the experiment for each of the 42 age groups into which the measurements were divided. These they find to be quite small even though they are here and there significant, and they argue that the additional height and weight of the "controls" was without effect on the comparison of subsequent growth.

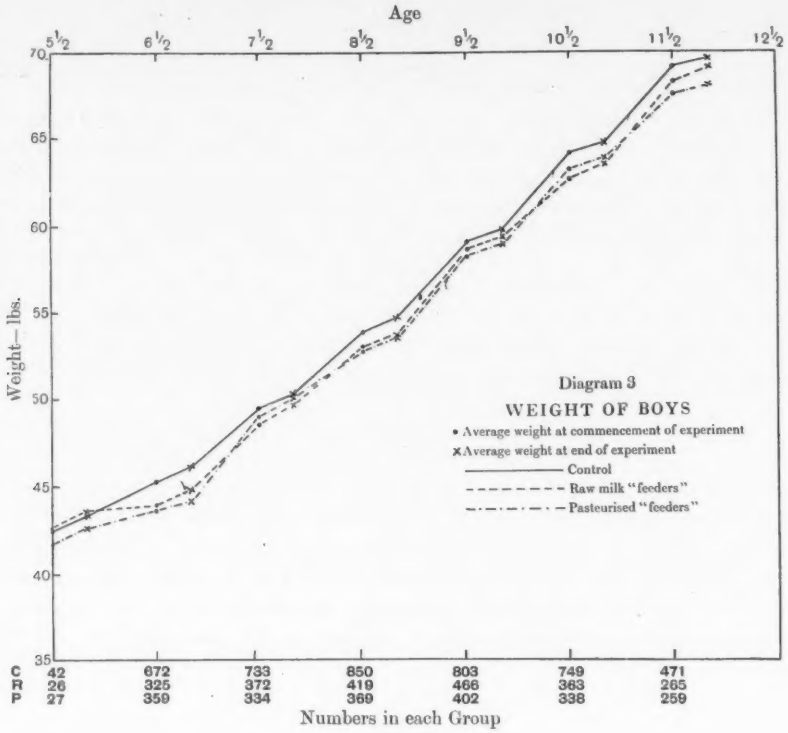
Now this might have been a perfectly good argument had the height and weight been selected directly, but if, as I have indicated was very likely the case, the selection was made according to some unconscious scale of well being, then it is surely natural to suppose that the relatively ill nourished "feeders" would benefit more than their more fortunate school mates, the "controls," would have done by the extra $\frac{3}{4}$ pint of milk per day.

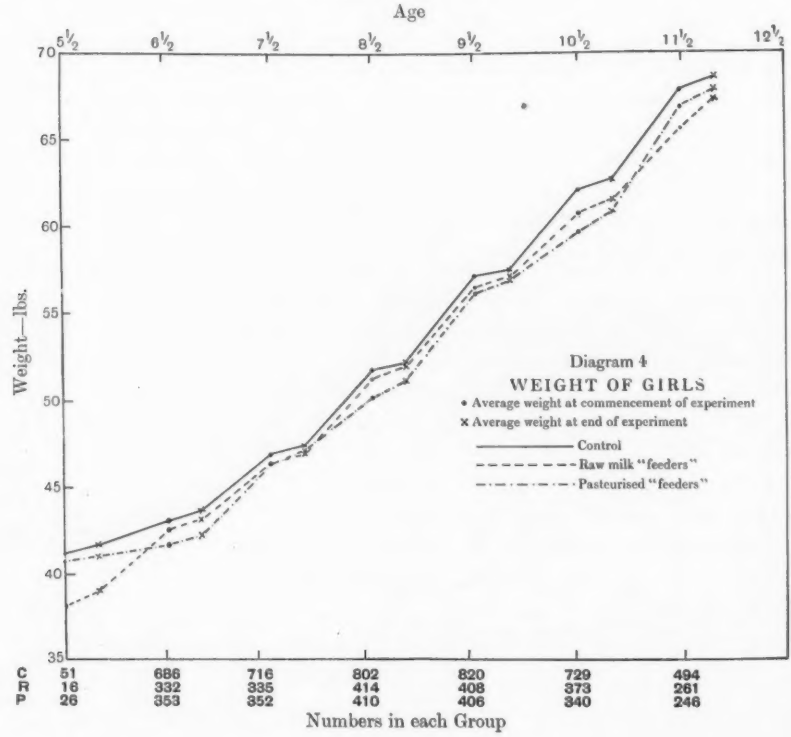
That being so how are we to regard the conclusions of the Report*:

(1) "The influence of the addition of milk to the diet of school children is reflected in a definite increase in the rate of growth both in height and weight."

This conclusion was probably true; the average increase for boys' and girls' heights was 8 per cent. and 10 per cent. over "controls" and for boys' and girls' weights was 30 per cent. and 45 per cent., respectively, and though, as pointed out,

* See footnote on p. 398.


Age
5½
6½
7½
8½
9½
10½
11½
12½
70
65
60
55
50
45
40
35
Weight—lbs.
Diagram 3
WEIGHT OF BOYS
• Average weight at commencement of experiment
× Average weight at end of experiment
Control
Raw milk "feeders"
Pasteurised "feeders"
C 42 672 733 850 803 749 471
R 26 325 372 419 466 363 265
P 27 359 334 369 402 338 259
Numbers in each Group


Age
5½
6½
7½
8½
9½
10½
11½
12½
70
65
60
55
50
45
40
35
Weight—lbs.
Diagram 4
WEIGHT OF GIRLS
• Average weight at commencement of experiment
× Average weight at end of experiment
Control
Raw milk "feeders"
Pasteurised "feeders"
C 51 686 716 802 820 729 494
R 16 332 335 414 408 373 261
P 26 353 352 410 406 340 246
Numbers in each Group

the figures for weights were wholly unreliable it is likely enough that a substantial part of the difference in height and a small part of that in weight were really due to the good effect of the milk. The conclusion is, however, shifted from the sure ground of scientific inference to the less satisfactory foundation of mere authority and guesswork by the fact that the "controls" and "feeders" were not randomly selected.

(2) "There is no obvious or constant difference in this respect between boys and girls and there is little evidence of definite relation between the age of the children and the amount of improvement. The results do not support the belief that the younger derived more benefit than the older children. As manifested merely by growth in weight and height the increase found in younger children through the addition of milk to the usual diet is certainly not greater than, and is probably not even as great as, that found in older children."

Now from the authors' point of view, believing in the validity of their comparisons in weight, this is much understating the case, as the following table derived from Capt. Bartlett's condensed tables shows:

Age in years	Gain in weight in ozs. by Feeders over Controls		Gain in height in inches by Feeders over Controls		As °/₀ of control			
					Weight		Height	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
5, 6 and 7	1·13 ± ·73	1·24 ± ·72	·083 ± ·011	·059 ± ·011	9	13	11	8
8 and 9	3·15 ± ·68	4·47 ± ·67	·071 ± ·011	·098 ± ·010	30	51	10	14
10 and 11	5·21 ± ·85	7·88 ± ·79	·037 ± ·012	·055 ± ·012	78	73	5	8

Note that the P.E.'s are calculated from Capt. Bartlett's tables and are subject, as his are, to his having interpreted the methods of the original Report correctly.

From this they might have concluded:

(*a*) That in the matter of weight older children, both boys and girls, derived more benefit than younger, while

(*b*) In height the younger boys did better than the older, though the difference is not quite significant, but that there was no regular tendency in the matter of girls' height.

In the light of previous criticism, however, we must be content to say that apparently the differential shedding of clothes between the "feeders" and the more fortunate "controls" is more marked with older children (and possibly with girls than with boys), and that there is some probability that younger boys gain in height more than older.

Finally, conclusion (3) runs: "In so far as the conditions of this investigation are concerned the effects of raw and pasteurised milk on growth in weight and height are, so far as we can judge, equal."

This conclusion has been challenged by Capt. Bartlett*, and by Dr Fisher and Capt. Bartlett†, who conclude that there is definite evidence of the superiority of raw over pasteurised milk in both height and weight.

Even they however point out that the raw and pasteurised milk were not supplied to the same schools, and their conclusion amounts to saying: "If the groups of children taking raw and pasteurised milk respectively were random samples from the same population, the observed differences would be decisively in favour of the raw milk."

Unfortunately they were not random samples from the same population: they were selected samples from populations which may have been different, and moreover the "controls" with which they were compared were not appropriate to either group; and so—again it is a matter of guess and authority—I would be very chary of drawing any conclusion from these small biased differences.

That is not to say that there is no difference between the effect of raw and pasteurised milk—personally I believe that there is and that it is in favour of raw milk—but that this experiment, in spite of all the good work which was put into it, just lacked the essential condition of randomness which would have enabled us to prove the fact.

This note would be incomplete without some constructive proposals in case it should be considered necessary to do further work upon the subject, and accordingly I suggest the following:

(1) If it should be proposed to repeat the experiment on the same spectacular scale,

(*a*) The "controls" and "feeders" should be chosen by the teachers in pairs of the same age group and sex, and as similar in height, weight and especially physical condition (i.e. well or ill nourished) as possible, and divided into "controls" and "feeders" by tossing a coin for each pair. Then each pair should be considered to be a unit and the gain in weight and height by the "feeder" over his own "control" should also be considered as a unit for the purpose of determining the error of the gain in weight or height.

In this way the error will almost certainly be smaller, perhaps very much smaller, than if calculated from the means of "feeders" and "controls."

If in addition the social status of each pair be noted (well to do, medium, poorly nourished or some such scale) further useful information will be available for comparing pasteurised and raw "feeders."

If this is found to be too difficult a perfectly good comparison can be made by adhering to the original plan of the 1930 experiment and drawing lots to decide which should be "controls" and which "feeders" (this is better than an alphabetical arrangement), but the error of the comparison is likely to be larger than in the plan outlined above.

* "Nutritional Value of Raw and Pasteurised Milk," by Stephen Bartlett, M.C., B.Sc. (*Journal of the Ministry of Agriculture*, April, 1931).

† *Nature*, April 18th, 1931, p. 591, "Pasteurised and Raw Milk."

(*b*) If it is at all possible each school should supply an equal number of raw and pasteurised "feeders," again by selection of similar children followed by coin tossing, but I fear that this is a counsel of perfection.

(*c*) Some effort should be made to estimate the weight of clothes worn by the children at the beginning and end of the experiment: possibly the time of year could be chosen so that there would be little change in this respect.

(2) If it be agreed that milk is an advantageous addition to children's diet—and I doubt whether any one will combat that view—and that the difference between raw and pasteurised milk is the matter to be investigated, it would be possible to obtain much greater certainty at an expenditure of perhaps 1—2 per cent. of the money* and less than 5 per cent. of the trouble.

For among 20,000 children there will be numerous pairs of twins; exactly how many it is not easy to say owing to the differential death rate, but, since there is about one pair of twins in 90 births, one might hope to get at least 160 pairs in 20,000 children. But as a matter of fact the 20,000 children were not all the Lanarkshire schools population, and I feel pretty certain that some 200—300 pairs of twins would be available for the purpose of the experiment.

Of 200 pairs some 50 would be "identicals" and of course of the same sex, while half the remainder would be non-identical twins of the same sex.

Now identical twins are probably better experimental material than is available for feeding experiments carried out on any other mammals, and the error of the comparison between them may be relied upon to be so small that 50 pairs of these would give more reliable results than the 20,000 with which we have been dealing.

The proposal is then to experiment on all pairs of twins of the same sex available, noting whether each pair is so similar that they are probably "identicals" or whether they are dissimilar.

"Feed" one of each pair on raw and the other on pasteurised milk, deciding in each case which is to take raw milk by the toss of a coin.

Take weekly measurements and weigh without clothes.

Some way of distinguishing the children from each other is necessary or the mischievous ones will play tricks. The obvious method is to take finger-prints, but as this is identified with crime in some people's minds, it may be necessary to make a different indelible mark on a fingernail of each, which will grow off after the experiment is over.

With such comparatively small numbers further information about the dietetic habits and social position of the children could be collected and would doubtless prove invaluable.

The comparative variation in the effect in "identical" twins and in "unlike" twins should furnish useful information on the relative importance of "Nature and Nurture."

* This is a serious consideration: the Lanarkshire experiment cost about £7500.

To sum up: The Lanarkshire experiment devised to find out the value of giving a regular supply of milk to children, though planned on the grand scale, organised in a thoroughly business-like manner and carried through with the devoted assistance of a large team of teachers, nurses and doctors, failed to produce a valid estimate of the advantage of giving milk to children and of the difference between raw and pasteurised milk.

This was due to an attempt to improve on a random selection of the controls which in fact selected as controls children who were on the average taller and heavier than those who were given milk.

The hypothesis is advanced that this was due not to a selection of the shorter, lighter children as such to take the milk, but to an unconscious bias leading the teachers to pick out for this purpose the needier children whom the milk would be most likely to benefit.

This hypothesis is supported by the fact that while the advantage derived from the milk was only 8—10 per cent. of the gain in height, without much variation for age, it was 30—45 per cent. of the gain in weight, varying from 9—13 per cent. in the younger children (who do not seem to have shed much clothing in the summer) up to 73—78 per cent. in the older children—who obviously did.

Suggestions are made for the arrangement:

(1) Of a similar large scale experiment on random lines, and

(2) Of a much smaller and cheaper experiment carried out on pairs of twins of like sex.

The second is likely to provide a much more accurate determination of the point at issue, owing to the possibility of balancing both nature and nurture in the material of the experiment.

ON THE "z" TEST.

By "STUDENT."

In the last number of *Biometrika* Professor Pearson sounds a warning note against the use of "Student's" z to determine the significance of an average difference between two sets of correlated variables.

As this use is one to which I attach considerable importance, and as Professor Pearson's criticism does not all seem to me to be concerned with my own method of using z, I should like to present the case from the point of view of the experimenter who for some reason or another has to work with small samples.

In the great majority of experiments of this kind we are concerned with a *difference*: e.g., of yield between two varieties of a cereal; of weight between pigs fed on complete food and others on food deficient in vitamins; of size of loaf between breads baked from different flours; of reaction times between alcoholised and non-alcoholised persons, and so on.

Now it is an elementary principle in all such experiments to reduce the error of such differences by arranging that they should lie between variates which, apart from the experiment, are as similar as possible.

Thus each pair of cereal plots should be grown not only on the same field but as near as possible to each other in that field; the pigs should be of the same sex, from the same litter and as nearly as possible the same weight at the start; the loaves should be mixed at the same time and put next to one another in the oven, and the alcoholised and non-alcoholised persons should alternate their rôles so as to compare each person with himself.

In other words, every care should be taken when planning the experiment to get the correlation between corresponding variates as high as possible, with the object of reducing the error and so of obtaining significant results from the small number of experiments which it is possible to carry out.

Now Professor Pearson's criticism may be summarised under three heads:

(1) That, assuming the advisability of the "z" Test, it is only when the value of z which is obtained is high that we can draw any useful conclusion: if it is low it cannot detect samples which may be abnormal in other ways.

Agreed, but this inability to detect abnormalities extraneous to the test itself is shared with all single tests of significance and the result is that the wise man will never go further in the direction of asserting similarity than to say, "The sample affords no evidence that, etc."

(2) That, since we cannot deduce with accuracy the correlation of the population at large from the small sample before us, we are debarred from making use of that correlation to reduce our error.

But in fact we do *not* use the correlation in testing the significance. In obtaining our sample of differences, yes, but once we have the differences they are merely a sample from the indefinitely large population of differences, which might have been produced under similar conditions and I may say with the same ρ (not r) and which we, rightly or wrongly, assume to be normal. At all events Professor Pearson is not here attacking z on the grounds of lack of normality. The correlation will vary from sample to sample, just as does the mean or standard deviation, but these variations in correlation do not affect the fact, which Professor Pearson admits, that in a normal population z can be used to test the significance of the mean of small samples.

What we actually ask ourselves is the following question:

If the average difference between A and B in the population were zero, what would be the probability of obtaining a sample of differences giving a value of z as high as that observed? and if this probability is sufficiently small we say that the difference is significant.

(3) Professor Pearson warns us to be careful to draw our conclusion from the experiment we have carried out and according to the particular set of differences which we have tested for significance.

In this of course I agree with him, yet I do not feel that the warning is very much needed. In the hyoscyamine experiment which he quotes, we are able to deduce significance from a consideration of the effects of these drugs on the same individuals while we could not do so from groups of different individuals. But surely no one is much interested in the latter point; if I am to take one of the drugs I will pay a good deal of attention to the probability that lævo will make me personally sleep longer than dextro and very little to the fact that the experiments give no satisfactory answer to the question of what will happen if I take lævo and my wife dextro.

To sum up, in properly planned experiments errors should be reduced as much as possible by the selection of highly correlated individuals to compare with one another. This correlation should to a greater or lesser extent reduce the variation in the differences between these individuals but does not prevent them being considered to be a sample drawn from a population of differences to which the "z" Test may be applied. Finally, care must be taken in planning the experiment that the differences to be examined for significance shall be those which furnish an answer to the question which we are asking.

Further Remarks on the "z" Test.

(1) I am afraid I have not put my difficulties as to the "z" Test clearly, for "Student" does not seem to me to throw light on them. I can illustrate this best from his words cited below.

"In the hyoscyamine experiment which he (i.e. K. P.) quotes we are able to deduce significance from a consideration of the effects of these drugs on the same individuals, while we could not do so from groups of different individuals. But surely no one is much interested in the latter point; if I am to take one of the drugs I will pay a good deal of attention to the probability that lævo will make me personally sleep longer than dextro, and very little to the fact that the experiments give no satisfactory answer to the question of what will happen if I take lævo and my wife dextro."

Now apart from the question of whether it is not important for the medical profession to know whether lævo or dextro will be more effective on the population at large, "Student" seems to me to misinterpret the outcome of his own test. It is true he operates with the two drugs on the same individual, but the actual answer to his problem is the probability, or improbability, that the one drug is more effective than the other on the *population at large*. Just the point of importance to the doctor, but which "Student" appears to have no interest in.

We put aside in the first place the influence of want of normality. We will suppose x and y to be two variates distributed normally with means $\bar{x}$, $\bar{y}$, standard deviations σ_x, σ_y and correlation r which may be zero, if the variates are so selected as to be independent. From this a small sample n may be taken in two different ways:

(a) By picking out n values of x, and n values of y and pairing them absolutely at random, or:

(b) By picking out n correlated values of x and y at random.

We will suppose in our sample that m_x and m_y are the means, s_x and s_y the standard deviations, and ρ the correlation. We can pass from (a) to (b) by putting $r=0$, but in doing so we change the value of $m_x - m_y$, for in (b) these are correlated with correlation r, and if in our sampling we get a high m_x we shall probably get a high m_y. Similarly s_1^2 and s_2^2 are correlated in (b) with correlation r^2, and again s_1^2, s_2^2 are correlated with ρ, the correlation being $r/\sqrt{2}$. To fix our ideas we may consider a population of Fathers and Sons for a character, say stature, of correlation 0·5 between Father and Son. Then the two problems we are considering are:

(a) Is the younger generation taller than the older, or not?

(b) What is the probability that the individual Son will be taller than his own Father?

If the sample be large there is no difficulty about answering these problems, but if the sample be small, we are supposed to get the best answer from the z test.

(2) I take it that there is no doubt in "Student's" mind about what z stands for in the problem. It is sometimes rather loosely said that if x and y are distributed normally then their difference is distributed normally. This is true, but it is better to say that $u=(x-\bar{x})-(y-\bar{y})$, where mean $\bar{u}=0$, is distributed normally with a standard deviation σ_u given by

$$\sigma_u^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y,$$

clearly $m_u = m_x - \bar{x} - m_y + \bar{y}$.

We have then for the distribution of u the curve

$$v = v_0 e^{-\frac{1}{2}u^2/\sigma_u^2},$$

and in samples n, m_u, s_u, we have m_u and s_u distributed by the independent curves

$$v' = v_0' e^{-\frac{1}{2}nm_u^2/\sigma_u^2}, \text{ and } v'' = v_0'' e^{-ns_u^2/\sigma_u^2}\left(\frac{s_u}{\sigma_u}\right)^{n-2}$$

which lead us directly to the distribution of

$$z = \frac{m_u}{s_u} = \frac{m_x - \bar{x} - m_y + \bar{y}}{\sqrt{s_x^2 + s_y^2 - 2\rho s_x s_y}}$$

by "Student's" curve

$$v''' = v_0'''/(1+z^2)^{\frac{1}{2}n}.$$

It is well to emphasise exactly what "Student's" curve does; it provides merely *one* measure* of the rarity of samples with a z equal or greater than the value observed in the sample. But to measure that rarity we require an *a priori* knowledge of the values of $\bar{x}$ and $\bar{y}$.

Now what are these $\bar{x}$'s and $\bar{y}$'s? They are the means of the *general* populations of x's and y's and have nothing whatever to do with the correlation of x and y, they are merely means of marginal totals. In our illustration they are the mean statures of the generation of Fathers and of the generation of Sons. In the hyoscyamine problems they are the average hours of sleep of a general population of men under the lævo-form and a general population of men under the dextro-form, quantities which the medical profession, if not "Student," are likely to be interested in. Now all we know is the result of a *small* sampling which provides in case (*a*) m_x, m_y, s_x and s_y, and in case (*b*) the additional quantity ρ.

Now if this be really all we know it is difficult to see how we can get further without making hypotheses as to $\bar{x}$ and $\bar{y}$, the very quantities we wish to find. Let us consider illustrations: (i) We are dealing with the correlation table of pairs of adult brothers. Here we *know* that $\bar{x} = \bar{y}$. Suppose z comes out large. We do not argue that $\bar{x}$ is not $= \bar{y}$, because we know it is. We merely say we have been unfortunate and have drawn a rare random sample. (ii) We are dealing with a sample from a correlation table of Fathers and Sons. Suppose z comes out as large as in the previous illustration, shall we say that we have again drawn a rare random sample, or that its rarity is due to $\bar{x}$ not being equal to $\bar{y}$? Surely we are thrown back on measuring in some way the relative magnitudes of the improbability of drawing a rare random sample, and the improbability of the mean stature of Sons not equalling that of Fathers. Now it appears to me that "Student," in applying his test, is like one who is constantly repeating to himself: "It is more probable that the mean stature of Fathers is not equal to that of Sons, than that I should draw a rare sample."

A priori, with no knowledge of Fathers and Sons, but that of a small sample, we should have no numerical measure of the improbability of their statures being

* It is the rarity of a *ratio* not of any of the *absolute* quantities m_x, m_y, s_x, s_y, and ρ provided by the experiment. I have never been able to conceive why testing one character is better than testing three.

unequal. Given that the Sons are adult, we might argue that Fathers with adult sons have begun to shrink in stature with age, and therefore have a stature less than that of Sons; or we might argue that Fathers have lived to have adult sons, that many of their Sons will probably not live, owing to their physique, to marry and have adult sons; thus the Fathers will be likely to be stronger and probably bigger men than their average Sons. The truth or falsehood of such hypotheses cannot be determined *a priori*; above all the statistician should not fabricate hypotheses of which he has not measured the relative probability. Even if we think that it is more reasonable to suppose $\bar{x}$ not equal to $\bar{y}$, than to believe that we have drawn a somewhat rare sample, we are confronted by the difficulty that we do not know whether to take $\bar{x}$ greater or less than $\bar{y}$. Obviously if we take it one way we shall make the sample more probable; if we take it the other way the sample becomes more improbable than before. It clearly is not enough to say the sample indicates that the mean statures of Fathers and Sons are not equal, or that the mean hours of sleep produced by the lævo- and dextro-forms are different; we wish to know which is the greater, which is the more efficient. In the lack of *a priori* knowledge—such as, of course, exists in the case of Fathers and Sons—we can only appeal to our experiment; we suppose (or I suppose "Student" supposes) that if m_x be $> m_y$, then $\bar{x}$ will probably be $> \bar{y}$. To suppose this is to make z smaller and accordingly the sample appear more probable. In other words a choice is made of $\bar{x}$, $\bar{y}$ which makes the sample more probable. Let us remember that if $\bar{x}$ were $> \bar{y}$, it would still be possible to draw a *small* sample in which m_x was less than m_y. For example, in picking out ten Fathers at random it would be easy to find one in which they were taller than their ten Sons—although the mean Father is really shorter than the mean Son; we should reach a fallacious conclusion as to the sign of $\bar{x}-\bar{y}$ by supposing it to be the same as that of m_x-m_y, in other words the hypothesis by which we have made the sample more probable was an exceedingly improbable one as to $\bar{x}$ and $\bar{y}$. Thus the hypothesis that my sample is too rare to have been drawn has led me to make a more improbable hypothesis as to $\bar{x}$, $\bar{y}$.

But if we are to give $\bar{x}-\bar{y}$ such a sign as to make the value of z more probable, why not choose them so as to give the probability of the sample its maximum value? Something like this has been done on other occasions. We shall make the sample most probable if we make $z=0$, or if $\bar{x}-\bar{y}$ be taken equal to m_x-m_y! If there be no correlation between x and y, we see that the most reasonable hypothesis is that $\bar{x}=m_x$ and $\bar{y}=m_y$, thus the "z" Test appears to land us in the old-fashioned conclusion, that failing a knowledge of $\bar{x}$ and $\bar{y}$ the most probable result is that they are equal to m_x and m_y. If the two variates be correlated, say Father and Son, then if we have found m_x from n Fathers, the most probable value of m_y will be

$$m_y=\bar{y}+\frac{r\sigma_y}{\sigma_x}(m_x-\bar{x}),$$

and if we suppose we have got that value, then

$$z=\frac{m_x-m_y-\bar{x}+\bar{y}}{\sqrt{s_x^2+s_y^2-2\rho s_x s_y}}$$

may be taken as nearly

$$= \frac{(m_x - \bar{x})\left(1 - \frac{r\sigma_y}{\sigma_x}\right)}{\sqrt{s_x^2 + s_y^2 - 2\rho s_x s_y}},$$

and we shall get the most reasonable value of the sample by taking $\bar{x} = m_x$ and therefore from the probable value of m_y above, $\bar{y} = m_y$.

In general therefore the "z" Test—if we proceed on the hypothesis, which "Student" appears to use, that we have got to make the sample more probable by some assertion as to $\bar{x}$ and $\bar{y}$—leads to the conclusion that the best we can do is to suppose $\bar{x} = m_x$ and $\bar{y} = m_y$.

This seems to me the logical conclusion of the test in question, when we assume that a high value of z denotes that $\bar{x}$ has one or another relation to $\bar{y}$. In our ignorance as to $\bar{x}$ and $\bar{y}$, as must be the case with only a single small sample, we cannot tell how improbable or probable any hypothesis as to $\bar{x}$ and $\bar{y}$ may be. There is the improbability or probability of the sample, and the improbability or probability of our hypothesis as to $\bar{x}$ and $\bar{y}$, both are contributing to the value of z, and in ignorance as to what is likely as to $\bar{x}$ and $\bar{y}$ there does not seem to me any ground for throwing the improbability of the sample *wholly* on to the relation of $\bar{x}$ and $\bar{y}$. If, in our ignorance of the latter, we are to give them values which make z more probable, then the best we can do is to take the values of $\bar{x}$ and $\bar{y}$, those which make z most probable; but those values may be such that wider research may show them to be highly improbable, or even impossible. I do not doubt the value of the "z" Test when $\bar{x}$ and $\bar{y}$ are known, but, at present, I am unconvinced that with *small* samples it is an effective means of ascertaining whether $\bar{x}$ or $\bar{y}$ is the greater.

"Student" appears to think the "z" Test will answer the question whether on the average the lævo-form will produce more or fewer hours of sleep than the dextro-form on the *same* individual. It seems to me that if we follow him, accepting his method, we test whether $\bar{x}$ is greater or less than $\bar{y}$, i.e. we test whether one form is more effective than the other on what constitute two general populations, and we turn back and argue from this as to what is likely to happen when we administer the two forms of the drug to the same individual. There can be no difference in $\bar{x}$ and $\bar{y}$ in the two cases (a) and (b) as "Student's" words would lead one to think.

"Student" believes in working with highly correlated variates, and suggests the higher the correlation the more *trustworthy* will be the result. As I have indicated, where there is high correlation, the value of z will probably approach the form

$$\frac{(m_x - \bar{x})\left(1 - \frac{r\sigma_y}{\sigma_x}\right)}{\sqrt{s_x^2 + s_y^2 - 2\rho s_x s_y}},$$

or, if σ_x be nearly σ_y and $r \to 1$ since then ρ will $\to 1$), the form:

$$\frac{(m_x - \bar{x})(1 - r)}{s_x - s_y}.$$

In other words z will tend to be the ratio of the product of two small quantities to a third small quantity. The value of z will therefore fluctuate badly from one *small* sample to another, because the probable value of both of these quantities will be zero. I have discussed these fluctuations of z elsewhere. But I am by no means certain that they are not greater when $r \rightarrow 1$ than when $r \rightarrow 0$. To take x and y for a *few* highly correlated cases may, as "Student" suggests, remove certain sources of "error," but at the same time it removes certain sources of difference in the general population, which may cause, for example, two drugs to act on the whole population differently. Perhaps this is only hinting that small samples can rarely give us really accurate information. For reasons such as the above I seriously doubt whether "Student's" suggestion that an experiment with 50 pairs of like twins in the use of raw and pasteurised milk*, however interesting it might be, would give "more reliable results," than could be obtained from the 20,000 children of the Lanarkshire experiment. The latter experiment was no doubt from the statistical standpoint carried out ineffectually. But as "Student" tells me he proposes to attack the twin problem by the z method, he will have to make an assumption as to $\bar{x}$ and $\bar{y}$, the very quantities he wants to measure, which may turn out to be highly improbable, when their values on a large population are satisfactorily determined.

The above criticism of "Student's" identical twin proposal depends solely on the application of the z method to the data. I leave aside the question whether a selected population of only 50 pairs of twins can be fairly taken to represent in their method of growth and their physical constitution the general population of children of a great variety of constitutions and ages. I see no difficulty, considering the large number of children dealt with, in selecting from the original schedules, enough children of each class—control, raw milk feeders, pasteurised milk feeders—to obtain equality in the initial conditions, and thus to rectify the "fatal mistake" which "Student" holds vitiates the original inquiry.

By the kindness of Miss F. N. David, I have been provided with random samples of 10 from a population of Mothers and Daughters with regard to stature. The samples were extracted in the usual manner by aid of Tippett's Random Numbers. Actually with regard to the entire population of the schedules, there is a difference in stature of about 1″·4 between Mother and Daughter to the advantage of the latter, and this is about twenty times the probable error of the difference, and undoubtedly significant. The correlation of their statures is 0·507. But we are supposed to know nothing of this, and we propose by a single small sample to learn, if we can, something about these statures. As we do not know whether Mother or Daughter is to be considered the taller, it seems not unreasonable to take the limit of significance of "Student's" z to be unity. Two experiments were made (a) by selecting independently and at random a woman from the older and one from the younger generation, i.e. not mother and daughter. The ten samples gave means which when averaged showed a difference of 1″·41 between the two generations, as against the actual 1″·39, and a mean standard deviation of the

* See his paper, p. 398 of this volume.

difference of 3″·644 as against the actual 3″·539. The values for the mean difference and for the mean standard deviation were relatively steady. No single z, however, exceeded the limit, and, judging by any *single* case (and such a single case is "Student's" soporifics example), we should have to say either that the sample was not large enough to distinguish between statures of Mother and Daughter, or that there exists no difference between them for this character. The former would be the more reasonable statement.

In the second series (b) of ten experiments, the Mother was taken with her own Daughter. Here eight out of the ten samples showed no significance in the value of z and accordingly nothing could be learnt from them. In two cases z exceeded unity, and accordingly if "Student" had met with one of those cases, instead of one of the eight non-committal judgments, he would no doubt have argued as in the case of the single sample of the soporifics, that Daughters are taller than their Mothers. The actual mean difference was 1″·51, not as close as in (a), and the standard deviation of the difference 2″·432 as against the actual value in the whole population of 2″·409, a better result than in (a). But when the details are examined, we find a marked difference between (a) and (b). The standard deviations in (a) are relatively steady and have a coefficient of variation about the true value of 12·41, but those of (b) vary extravagantly, having a coefficient of variation of no less than 38·76, more than three times that of the series in (a). Now the essential factor which makes the difference in the distribution of the two sets of standard deviations is the correlation coefficient, and the variability of this for samples of ten is the source of the wide fluctuations of the standard deviation in (b). The standard deviation which forms the numerator of "Student's" z is by hypothesis independent of the difference of the means, therefore it might have occurred equally well with any other of the differences of means. In doing so, the lowest standard deviation would have made seven of the ten z's significant! The two z's which are significant owe their significance to the fact that they have the lowest standard deviations of all the differences. Had the standard deviation of the difference taken its actual value in the population at large not a single z would have shown significance. Had the difference of statures taken its actual value 1″·39 and the standard deviation of the difference its actual value 2″·409, we should have had for samples of ten, $z = \cdot 577$, which does not indicate significance. This is merely repeating the result of (a)—i.e. nothing to be learnt.

In fact the two significant values of z arise from the extreme variation of

$$\sqrt{s_x^2 + s_y^2 - 2\rho s_x s_y} \text{ from } \sqrt{\sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y}$$

owing to the correlation coefficient ρ being so variable in the case of samples like 10. No doubt further experimental sampling is desirable, but these few experiments seem to me to indicate that, there being no organic relation between the numerator and denominator of z, we may get by chance a large value of z, simply by obtaining a small value of its denominator, which stands in no relation to its numerator. This, as I have endeavoured elsewhere to indicate, is in accordance

with the extreme leptokurtosis of the z distribution for *small* samples. I have no objection to z in the case of large samples, say of 50 or more, where indeed it gives results which might for practical purposes be equally well obtained from the normal curve. But I think there is real danger of z being accepted as a solution of the difficulties which inevitably attach themselves to small samples. For this reason I should prefer the result for the uncorrelated material (a) above, i.e. "nothing to be learnt," to the chance provided by (b), that in two out of ten cases I might conclude that the Daughter was taller than the Mother, until my mind is relieved of the doubt that these two cases do not themselves arise from the chance distribution of the denominator, and have nothing to do with the inequality of $\bar{x}$ and $\bar{y}$ in the numerator. Take for illustration the highest value of z, 1·7450, in the ten (b) samples, and let us find how it arises. The actual difference between Mothers' and Daughters' statures in this case is nothing out of the way, it stands only third on the list of differences. The S.D. of Mothers in this sample is 2″·85 in *excess* of the value of Mothers in general 2″·39; the S.D. of Daughters in this sample is 2″·02, about as much in defect of 2″·61 as the Mothers are in excess. What then has caused z to reach its maximum value? It is the extreme smallness of the standard deviation of the difference.

This standard deviation has no relation to the numerator of z, and would have produced z's over unity in the case of seven of the ten samples, if it had occurred with them. With the standard deviations of the Mothers and of the Daughters in this sample being nothing out of the way, how does this small standard deviation of the difference arise? Solely because the value of ρ, the correlation in the sample, is +·9353 instead of the actual +·5070. The large z is due, not to any excessive value of $m_x - m_y$, but to something having no organic relation with it, namely, the excessive variations in correlation, which can easily arise in small samples. That is to say, we are going to assert that $\bar{x}$ cannot be equal to $\bar{y}$, not because m_x differs very widely from m_y, but because we have got a sample of Mothers and Daughters in a sample of ten with a correlation 80 % greater than its true value! If "Student" uses like twins with a correlation of at least the order +·9, he may easily in small random samples obtain correlations approaching unity, and in this case with s_x equal to, or approaching, s_y, almost any small difference of m_x and m_y will be found to give a most significant z. The result of this short series of experiments makes me doubt the wisdom of "Student's" choice of highly correlated material as suitable for his "z" Test. What he may gain by likeness of factors, he may more than lose in the variations of correlation which occur in such small samples.

KARL PEARSON.

MISCELLANEA.

(i) Historical Note on the Distribution of the Standard Deviations of Samples of any Size drawn from an indefinitely large Normal Parent Population.

EDITORIAL.

(1) The familiar distribution of the standard deviations of samples from an indefinitely large normal population appears to have been discovered on several occasions—without doubt independently—and by later writers attributed to various investigators. There can, we think, be no doubt that the original discoverer was Helmert and that he much antidates other claimants.

The memoirs in which he obtained it appear in Schlömilch's *Zeitschrift für Mathematik und Physik*, Bd. XX. S. 303—where he obtains the distribution of the second moment in samples taken about a *fixed* axis—and in the *Astronomische Nachrichten*, Bd. LXXXVIII. No. 2096, where he deduces from the result of the earlier paper the distribution of the standard deviations in the samples. The date of the former paper is 1875 and the latter 1876.

Some small excuse for the fact that Helmert has, especially by the English school of statisticians, been overlooked lies in his proof having so to speak been broken into two pieces, the second and more important section appearing in a journal which would be rarely consulted by statisticians. Set against this we must remark that Helmert's result has been reproduced in German textbooks on the theory of observations*.

Helmert's original method of deducing the now well-known equation

$$y = y_0 e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}} \Sigma^{n-2};$$

for the distribution of the standard deviation Σ in samples of size n from a normal population of standard deviation σ, differs considerably from more recent investigations; and we have accordingly thought it worth while to reproduce it here, combining what is requisite from his two memoirs, and but slightly altering his notation and arrangement.

(2) Let the parent population be $y = y_0 e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$, origin the mean, and let the n values x_1, $x_2 \ldots x_n$ to be drawn from this population supposed indefinitely large (or let each individual returned before a redrawing).

Then the chance of a sample lying between x_1 and $x_1 + dx_1$, x_2 and $x_2 + dx_2 \ldots x_n$ and $x_n + dx_n$ is given by

$$P = \text{constant} \times e^{-\frac{1}{2}\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{\sigma^2}} dx_1 dx_2 \ldots dx_n.$$

The mean of the sample $\bar{x}$ and its standard deviation Σ may be considered as given by

$$\bar{x} = (x_1 + x_2 + \ldots + x_n)/n,$$
$$\Sigma^2 = (x_1^2 + x_2^2 + \ldots + x_n^2)/n - \bar{x}^2,$$

the origin of all x's being the mean of the parent population. The x_1, x_2, ... x_n may be looked on as n independent variates.

* See in particular E. Czuber: *Theorie der Beobachtungsfehler*, Leipzig, 1891, S. 147 u.s.w.

Now let $\xi_s = x_s - \bar{x}$, then

$$\overset{n}{S}_1(\xi_s) = 0, \text{ and } \overset{n}{S}_1(\xi_s^2) = \overset{n}{S}_1(x_s^2) - n\bar{x}^2.$$

We now make a change of the independent variates $x_1, x_2 \dots x_n$ to the variates $\bar{x}, \xi_1, \xi_2 \dots \xi_{n-1}$, which are linearly connected with them. We have

$$x_1 = \xi_1 + \bar{x}, \quad x_2 = \xi_2 + \bar{x} \dots x_{n-1} = \xi_{n-1} + \bar{x}, \quad x_n = -(\xi_1 + \xi_2 + \dots + \xi_{n-1}) + \bar{x}.$$

Since the change is linear the Jacobian J will be a mere numeric independent of the variates and

$$dx_1 dx_2 \dots dx_n = J d\bar{x} d\xi_1 d\xi_2 \dots d\xi_{n-1}.$$

The J can be thrown into the constant and we have the chance of a sample lying between

$$\bar{x} \text{ and } \bar{x} + d\bar{x}, \quad \xi_1 \text{ and } \xi_1 + d\xi_1, \quad \xi_2 \text{ and } \xi_2 + d\xi_2 \dots \xi_{n-1} \text{ and } \xi_{n-1} + d\xi_{n-1},$$

given by

$$P = \text{constant} \times e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2} + \frac{n\bar{x}^2}{\sigma^2}} d\bar{x} d\xi_1 d\xi_2 \dots d\xi_{n-1}.$$

Since our variates are independent we can integrate out $\bar{x}$ from $-\infty$ to $+\infty$, and we find, modifying our constant, that the chance of the occurrence of the system

$$\xi_1 \text{ and } \xi_1 + d\xi_1, \quad \xi_2 \text{ and } \xi_2 + d\xi_2 \dots \xi_{n-1} \text{ and } \xi_{n-1} + d\xi_{n-1}$$

is given by

$$P = \text{constant} \times e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}} d\xi_1 d\xi_2 \dots d\xi_{n-1}.$$

Here Σ^2 included ξ_n^2, while ξ_n by virtue of the relation $\overset{n}{S}_1(\xi_s) = 0$ is equal to $-\overset{n-1}{S}_1(\xi_s)$.

Helmert now makes a further and very ingenious change of co-ordinates, again of a linear character. He takes

$$t_1 = \sqrt{\frac{2}{1}}\left(\xi_1 + \frac{1}{2}(\xi_2 + \xi_3 + \dots + \xi_{n-1})\right),$$

$$t_2 = \sqrt{\frac{3}{2}}\left(\xi_2 + \frac{1}{3}(\xi_3 + \xi_4 + \dots + \xi_{n-1})\right),$$

$$t_3 = \sqrt{\frac{4}{3}}\left(\xi_3 + \frac{1}{4}(\xi_4 + \xi_5 + \dots + \xi_{n-1})\right),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$t_s = \sqrt{\frac{s+1}{s}}\left(\xi_s + \frac{1}{s+1}(\xi_{s+1} + \xi_{s+2} + \dots + \xi_{n-1})\right),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$t_{n-1} = \sqrt{\frac{n}{n-1}}\,\xi_{n-1}.$$

The advantage of this system lies in the property that

$$\overset{n-1}{S}_1(t_s^2) = \overset{n}{S}_1(\xi_s).$$

* Actually

$$J = \begin{vmatrix} \frac{dx_1}{d\xi_1}, & \frac{dx_1}{d\xi_2}, & \dots & \frac{dx_1}{d\xi_{n-1}}, & \frac{dx_1}{d\bar{x}} \\ \frac{dx_2}{d\xi_1}, & \frac{dx_2}{d\xi_2}, & \dots & \frac{dx_2}{d\xi_{n-1}}, & \frac{dx_2}{d\bar{x}} \\ \dots\dots & \dots\dots & & \dots\dots & \dots\dots \\ \frac{dx_n}{d\xi_1}, & \frac{dx_n}{d\xi_2}, & \dots & \frac{dx_n}{d\xi_{n-1}}, & \frac{dx_n}{d\bar{x}} \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, \dots 0, & 1 \\ 0, & 1, & 0, \dots 0, & 1 \\ \dots\dots & \dots\dots & & \dots\dots \\ -1, & -1, & -1, \dots\dots & 1 \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, \dots & 1 \\ 0, & 1, & 0, \dots & 1 \\ \dots\dots & \dots\dots & & \dots\dots \\ 0, & 0, & 0, \dots & n \end{vmatrix} = n.$$

If we square the $(n-1)$ t's and add we find

$$\sum_1^{n-1}(t_s^2)=2(\xi_1^2+\xi_2^2+\ldots+\xi_{n-1}^2+S'(\xi_s\xi_{s'})),$$

where S' denotes the sum of the products of ξ_1, ξ_2 ... ξ_{n-1}, s not being equal to s'.

The reader will see that this result follows at once by squaring out and adding a few of the t's.

But since $\sum_1^n(\xi)=0$, we have

$$0=\xi_n^2-\xi_1^2-\xi_2^2-\ldots-\xi_{n-1}^2-2S'(\xi_s\xi_{s'}),$$

and adding to the above result we find

$$\sum_1^{n-1}(t_s^2)=\sum_1^n(\xi_s^2)=n\Sigma^2.$$

Hence the chance that a sample will give t's lying between

$$t_1 \text{ and } t_1+dt_1,\quad t_2 \text{ and } t_2+dt_2 \ldots t_{n-1} \text{ and } t_{n-1}+dt_{n-1}$$

is

$$P=\text{constant}\times e^{-\frac{1}{2}\sum_0^{n-1}(t_s^2)/\sigma^2}\,dt_1dt_2\ldots dt_{n-1}.$$

The constant has been modified by the introduction of a Jacobian J', but as the transformation is linear this is a mere numeric*.

Now the element of volume $dt_1dt_2\ldots dt_n$ has for its integrand only a function of $t_1^2+t_2^2+\ldots+t_{n-1}^2$, i.e. the radius squared of a hypersphere in $n-1$ dimensional space. If we call the radius of this sphere R the integrand is $e^{-\frac{1}{2}R^2/\sigma^2}$, and we may replace the element of volume, $dt_1dt_2\ldots dt_{n-1}$, by the volume of the spherical shell between R and $R+dR$, this is $R^{n-2}dR\times G$, where G is a numeric depending on n, which may be thrown again into the constant.

Accordingly the chance of a sample lying between R and $R+dR$ is

$$P=\text{constant}\times e^{-\frac{1}{2}\frac{R^2}{\sigma^2}}R^{n-2}dR,$$

but $R=\sqrt{n}\Sigma$, and accordingly with a final change of constant the chance that a sample will have a standard deviation lying between Σ and $\Sigma+d\Sigma$ is

$$P=\text{constant}\times e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}}\Sigma^{n-2}d\Sigma,$$

that is the required frequency distribution of Σ is given by

$$y=y_0e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}}\Sigma^{n-2}.$$

To determine y_0 we integrate $y\,d\Sigma$ from 0 to ∞ and this gives N the total number of samples:

$$N=\int_0^\infty y\,d\Sigma=y_0\int_0^\infty e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}}\Sigma^{n-2}d\Sigma.$$

Put $v=\frac{1}{2}n\Sigma^2/\sigma^2$, and we have the integral reduced to a complete Γ-function, hence we deduce

$$y=\frac{Nn^{\frac{1}{2}(n-1)}}{2^{\frac{1}{2}(n-3)}\sigma^{n-1}\Gamma(\frac{1}{2}(n-1))}e^{-\frac{1}{2}\frac{n\Sigma^2}{\sigma^2}}\Sigma^{n-2},$$

which differs only in notation from Helmert's form in the *Astronomische Nachrichten*, Bd. LXXXVIII. S. 122, 1876.

We suggest that in future this equation should be referred to as Helmert's Equation.

* It is generally most expeditious in investigations of this kind to leave the changing constant to be determined in its final form by simply integrating out.

(ii) The Game of Heads and Tails—Some Notes on M. Paul Lévy's Paper: "Nuove Formule Relative al Giuoco di Testa e Croce."

By E. C. FIELLER.

M. Lévy, in his paper* "Nuove formule relative al giuoco di testa e croce," investigates the behaviour of the gain of one of the players, by considering arithmetic triangles that are, in a sense, modifications of Pascal's Arithmetic Triangle. In a paper† written in ignorance of, and published just after, M. Lévy's work, I have investigated the more general problem of the Duration of Play by the same method. In its essentials this method goes back to De Moivre, and many of the results that he obtained have been rediscovered since. M. Lévy's formulae [2], [47], and [48], for example, will be found in De Moivre's work‡, although he gives no proof of them. I gave what I believed to be a fairly complete account of the history of the problem in § 5 of my paper, but since its publication I have learnt of an article by Whitworth§ which is of considerable interest. It contains what appears to be the earliest mention of the representation of sets of games by zig-zag paths, a representation that is as suggestive as it is elegant. Whitworth also arrives at the result given in equation (1) below; his proof, which depends upon summing a series, is hardly as simple as that indicated here.

Let us suppose that two players engage, for unit stakes at each throw, in a series of games of heads and tails. After n games, the net gain must differ from n by an even number. If we write against the point $(n-2p, -n)$ $(p=0, 1, \ldots n)$ the number of sets of n games leading to a net gain $n-2p$, we shall obtain the successive terms of the binomial

$$(1+1)^n.$$

Giving n the values 0, 1, 2, 3, ... in succession, we obtain in this manner the Arithmetic Triangle that M. Lévy calls D, and writes in the form

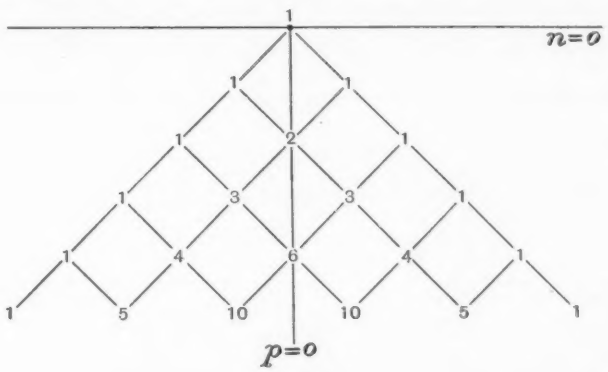


Diagram I.

It is clearly a rearrangement of Pascal's Arithmetic Triangle.

* *Giornale dell' Istituto Italiano degli Attuari*, Anno II. N. 2, April 1931, pp. 127—160.

† "The Duration of Play." *Biometrika*, Vol. XXII. Parts III. and IV. May 1931, pp. 377—404.

‡ *The Doctrine of Chances*, 3rd Edition, 1756. See Fieller, *loc. cit.* § 5.

§ "Arrangements of m Things of One Sort and n Things of another Sort under Certain Conditions of Priority." W. A. Whitworth, *Messenger of Mathematics*, Vol. VIII. 1878—79, pp. 105—114.

So long as no limit is set to the amount that either of the players may lose, the array D will represent the possibilities of the game. But if one of the players has initially only N stakes, the set must cease if his net loss ever reaches N; we can represent the possibilities of the game under these conditions by an array that M. Lévy calls $\Delta^{(N)}$. It is convenient to take the vertex of $\Delta^{(N)}$ at the point $(N, 0)$; we then build up $\Delta^{(N)}$ by writing at each point $(N+n-2p, -n)$ the number of sets of n games in the course of which the net loss never reaches N and at the end of which the net holding is $N+n-2p$. This is the number that M. Lévy calls $a_n^{p,(N)}$. Here $n=1, 2, 3, \ldots$ and, for each n, $p=0, 1, 2, \ldots t$, where t is the smaller of n and $\left[\frac{n+N}{2}\right]$.

Whitworth's geometrical representation associates with each sequence of games a zig-zag path of downward steps, each the diagonal of a unit square, the nth step going to the left or to the right, according as the nth game is lost or won. Such a series of steps we shall in what follows term simply *a path*. The entry at $(n-2p, -n)$ in D is the number of paths joining $(0, 0)$ and $(n-2p, -n)$; the entry at $(N+n-2p, -n)$ in $\Delta^{(N)}$ is the number of paths joining $(N, 0)$ and $(N+n-2p, -n)$ without coming into contact, on the way, with the vertical axis $N=0$. By establishing a one-one correspondence* between the remaining paths that join $(N, 0)$ and $(N+n-2p, -n)$, and the paths that join $(N, 0)$ and $(-N-n+2p, -n)$, we can show that, if $n-p \geqslant N > 2p-N$, their number is C^n_{p+N}. Altogether C^n_p paths join $(N, 0)$ and $(N+n-2p, -n)$. Accordingly

$$a_n^{p,(N)} = C^n_p - C^n_{p+N} \text{ if } n-p \geqslant N \geqslant 2p-N \quad \ldots\ldots(1).$$

The elements of $\Delta^{(N)}$ lying on the vertical axis are given by

$$a_n^{\frac{N+n}{2},(N)} = a_{n-1}^{\frac{N+n}{2}-1,(N)} \qquad (N+n \text{ even}) \quad \ldots\ldots(2);$$

for elements not covered by (1) and (2) we have

$$a_n^{p,(N)} = C^n_p \quad \ldots\ldots(3).$$

We can now see how to obtain directly any $\Delta^{(N)}$ from D. We take a D with vertex at $(N, 0)$ and, for $N+n-2p > 0$, subtract from the term of D lying at $(N+n-2p, -n)$, the term lying at $(-N-n+2p, -n)$. The accompanying figure indicates the formation of $\Delta^{(3)}$.

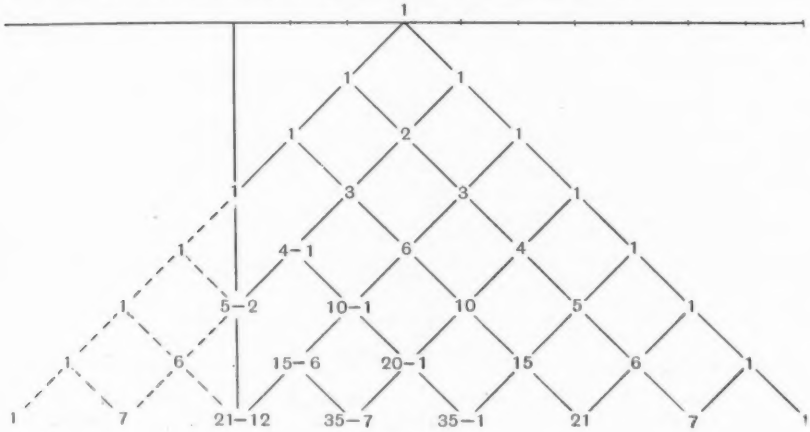


Diagram II.

* See Fieller, *loc. cit.* p. 379.

In writing down the terms of $\Delta^{(N)}$ that fall on the vertical axis, it is convenient to replace (2) by the equivalent result

$$a_n^{\frac{N+n}{2},(N)} = C_{\frac{N+n}{2}}^n - 2C_{\frac{N+n}{2}-1}^{n-1} \qquad (N+n \text{ even}) \ldots\ldots\ldots\ldots\ldots (2').$$

We may say, in short, that $\Delta^{(N)}$ is obtained by modifying the D with vertex at $(N, 0)$ by reflection in the vertical axis*. With the help of this result we can reach exact expressions for probabilities of which M. Lévy has given asymptotic values. Following M. Lévy, we denote by $P\{E\}$ the probability of an event E, by X the final gain, by Y and $-Y'$ the extreme values of the gain in the course of a set of games, and by Z the greater of $Y, -Y'$.

By the argument leading to equation (1), we have, immediately, for $0 \leqslant p \leqslant n$,

$$\begin{aligned} P\{X=n-2p,\ Y \geqslant N\} &= \frac{1}{2^n} C_{N+p}^n \qquad (n-N \geqslant p \geqslant \tfrac{1}{2}(n-N)) \\ &= \frac{1}{2^n} C_p^n \qquad (p \leqslant \tfrac{1}{2}(n-N)) \\ &= 0 \qquad (p > n-N) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4). \end{aligned}$$

Now if

$$\frac{n}{2} - r = \tfrac{1}{2} t \sqrt{n},$$

$$\frac{1}{2^n} C_r^n \sim \sqrt{\frac{2}{\pi n}}\, e^{-\frac{1}{2}t^2}.$$

Thus if $n-2p = x\sqrt{n}$, $N = y\sqrt{n}$, we have directly from (4) the asymptotic value given by M. Lévy†,

$$P\{X = x\sqrt{n},\ Y \geqslant y\sqrt{n}\} \sim \sqrt{\frac{2}{\pi n}}\, e^{-2\left(n-\frac{x}{2}\right)^2}.$$

If the final value of the gain be known, then instead of (4) we have, for the probability that the gain should have reached N in the course of the set,

$$\begin{aligned} P_x\{Y \geqslant N\} &= C_{N+p}^n / C_p^n \qquad (n-N \geqslant p \geqslant \tfrac{1}{2}(n-N)) \\ &= 1 \qquad (p \leqslant \tfrac{1}{2}(n-N)) \\ &= 0 \qquad (p > n-N); \end{aligned}$$

for the number of paths that can represent the course of the set is now reduced from 2^n to C_p^n.

Rewriting (4), we have

$$\begin{aligned} P\{X=n-2p,\ Y>N\} &= P\{X=n-2p,\ Y \geqslant N+1\} \\ &= \frac{1}{2^n} C_{N+p+1}^n \qquad (n-N-1 \geqslant p \geqslant \tfrac{1}{2}(n-N-1)) \\ &= \frac{1}{2^n} C_p^n \qquad (p \leqslant \tfrac{1}{2}(n-N-1)) \\ &= 0 \qquad (p > n-N-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5). \end{aligned}$$

Let $n = N+2s$. Then

$$P\{|X| > N\} = 2 \cdot \frac{1}{2^n}(C_{s-1}^n + C_{s-2}^n + \ldots + C_0^n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and from (5)

$$\begin{aligned} P\{Y>N\} &= \sum_{p=0}^{N+2s} P\{X=n-2p,\ Y>N\} \\ &= \sum_{p=0}^{s-1} P\{X=n-2p,\ Y>N\} + \sum_{p=s}^{2s-1} P\{X=n-2p,\ Y>N\} \\ &= 2 \cdot \frac{1}{2^n}(C_{s-1}^n + C_{s-2}^n + \ldots + C_0^n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7), \end{aligned}$$

so that, by (6) and (7), $P\{Y>N\} = P\{|X|>N\} \qquad (n = N+2s) \ldots\ldots\ldots\ldots\ldots\ldots (8).$

* Cf. Fieller, *loc. cit.* pp. 379, 384.

† Lévy, *loc. cit.* equation [35].

Let $n=N+2s+1$. Then

$$P\{|X|>N\}=2.\frac{1}{2^n}(C_s^n+C_{s-1}^n+\ldots+C_0^n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

and from (5)

$$\begin{aligned}P\{Y>N\}&=\sum_{p=0}^{N+2s+1}P\{X=n-2p,\ Y>N\}\\&=\sum_{p=0}^{2s}P\{X=n-2p,\ Y>N\}+\sum_{p=s+1}^{2s}P\{X=n-2p,\ Y>N\}\\&=\frac{1}{2^n}(C_s^n+C_{s-1}^n+\ldots+C_0^n)+\frac{1}{2^n}(C_{s-1}^n+C_{s-2}^n+\ldots+C_0^n) \quad \ldots\ldots\ldots\ldots(10),\end{aligned}$$

so that, by (9) and (10),

$$P\{Y>N\}=P\{|X|>N\}-\frac{1}{2^n}C_s^n \qquad (n=N+2s+1) \quad \ldots\ldots\ldots\ldots(11).$$

The connection of (8) and (11) with M. Lévy's [28] is clear. Since

$$\frac{1}{2^n}C_s^n\to 0 \text{ as } n(=N+2s+1)\to\infty,$$

we have M. Lévy's asymptotic result $P\{Y>N\}\sim P\{|X|=N\}$.

Writing $\underset{n}{P}\{E\}$ for the probability that E should happen in the course of n games, we have

$$\underset{N+2s+2}{P}\{Y>N\}=\underset{N+2s+1}{P}\{Y>N\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

since the gain cannot reach $N+1$ in the course of $N+2s+2$ games, if it has not done so in the course of the first $N+2s+1$ games.

We can transform equations (6), (7), (9) and 10 by means of the formula*

$$\text{Sum of 1st } s \text{ terms of } \{x+(1-x)^n\}=I_x(n-s+1,\ s)$$

$$=\int_0^x x^{n-s}(1-x)^{s-1}\,dx\Big/\int_0^1 x^{n-s}(1-x)^{s-1}\,dx \quad \ldots\ldots(13),$$

which expresses the sum of the end terms of a binomial as the proportional area under a Pearson's Type I curve.

Using (13) we have

$$\underset{N+2s}{P}\{|X|>N\}=\underset{N+2s}{P}\{Y>N\}=2I_{\frac{1}{2}}(N+s+1,\ s) \quad \ldots\ldots\ldots\ldots\ldots\ldots(14),$$

$$\underset{N+2s+1}{P}\{|X|>N\}=2I_{\frac{1}{2}}(N+s+1,\ s+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$$\underset{N+2s+1}{P}\{Y>N\}=\underset{N+2s+2}{P}\{Y>N\}=2I_{\frac{1}{2}}(N+s+2,\ s+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(16).$$

We return to M. Lévy's triangle Δ, which corresponds to sets played under the condition that the game shall cease if the gain vanishes. After the first game, the winner is in the position of a man beginning to play with fortune 1. Accordingly, the terms of Δ, after the initial term, are exactly the terms of a triangle $\Delta^{(1)}$ with vertex at $(1,\ -1)$, i.e. they are the terms of a Pascal's Triangle D with vertex at $(1,\ -1)$, modified by reflection in the vertical axis. In other words, we can (see Diagram III) obtain the portion of Δ that lies to the right of its central line by setting out a Pascal's Triangle D with its vertex at $(1,\ -1)$, and subtracting from each term thus written the term immediately to its right. Hence the term of Δ at $(n-2p,\ -n)$ is

$$a_n^p=C_p^{n-1}-C_{p-1}^{n-1} \qquad (n>2p) \quad \ldots\ldots\ldots\ldots\ldots\ldots(17);$$

this result is, of course, exactly the same as that given by M. Lévy†,

$$a_n^p=\frac{|n-2p|}{n}C_p^n,$$

* K. Pearson, *Biometrika*, Vol. XVI. p. 202.

† Lévy, *loc. cit.* equation [2].

but appears to be of a more convenient form. We deduce immediately from (17) that the probability of a final gain greater than $n-2p$ is

$$\frac{1}{2^n}\sum_{q=0}^{p-1} a_n^q = \frac{1}{2^n} C_{p-1}^{n-1} = \frac{1}{2^n}\frac{p}{n} C_p^n \quad \text{.....(18),}$$

which contains M. Lévy's equation [6] as a particular case. Putting $n-2p=x\sqrt{n}$, we have, from (18),

$$\frac{1}{2^n}\sum_{q=0}^{p-1} a_n^q \sim \frac{p}{n}\sqrt{\frac{2}{\pi n}}\, e^{-\frac{1}{2}x^2} \quad \text{.....(19).}$$

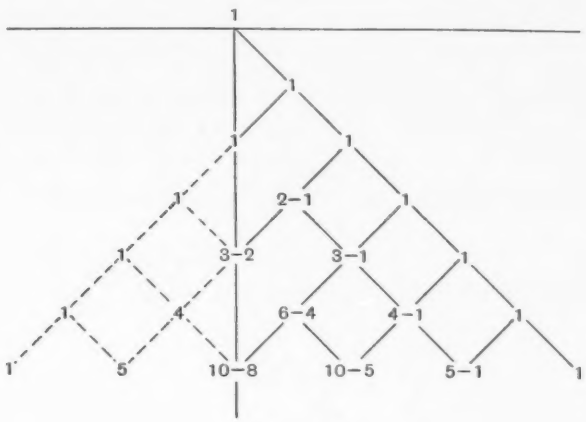


Diagram III.

The asymptotic value that M. Lévy indicates* appears to be that of the probability of a final gain *equal to or greater than* $n-2p$. The exact value of this probability is

$$\frac{1}{2^n}\sum_{q=0}^{p} a_n^q = \frac{1}{2^n} C_p^{n-1}$$

$$= \frac{1}{2^n}\frac{n-p}{n} C_p^n \sim \sqrt{\frac{2}{\pi n}}\, e^{-\frac{1}{2}x^2} \quad \text{.....(20).}$$

Denoting by $x\sqrt{n}$ the gain at the end of a set of n games, in the course of which the gain never vanishes, we have, by (19) and (20), for all values of $p < \frac{n}{2}$,

$$\frac{P\{x\sqrt{n} > n-2p\}}{P\{x\sqrt{n} = n-2p\}} \sim 0 \quad \text{.....(21),}$$

and similarly, for all values of $p' > \frac{n}{2}$,

$$\frac{P\{x\sqrt{n} < n-2p'\}}{P\{x\sqrt{n} = n-2p'\}} \sim 0 \quad \text{.....(22).}$$

* Lévy, *loc. cit.* equations [14] and [15].

(iii) Note on the Distribution of the Standard Deviation of Sets of Three Variates drawn at Random from a Rectangular Distribution.

By H. L. RIETZ.
(University of Iowa.)

The distribution function for the standard deviations of samples of two drawn from a continuous rectangular parent distribution was given by Rider* in 1929.

It is the object of the present note to derive the exact distribution function of the standard deviation of three variates drawn from a continuous rectangular distribution from 0 to a.

Let x, y, z be the set of observed variates drawn from a rectangular distribution from 0 to a. Represent (x, y, z) as the rectangular co-ordinates of a point P in three dimensional space (Fig. 1). Then the probability that a sample will fall into a prescribed element $dx\,dy\,dz$, where x, y, z are now variables each in the interval 0 to a, is

$$\frac{1}{a^3}dx\,dy\,dz. \tag{1}$$

We seek the distribution of the standard deviations of samples of three where

$$x+y+z=3m, \tag{2}$$

$$(x-m)^2+(y-m)^2+(z-m)^2=3s^2. \tag{3}$$

The plane MNR of Fig. 1 represents the equation (2).

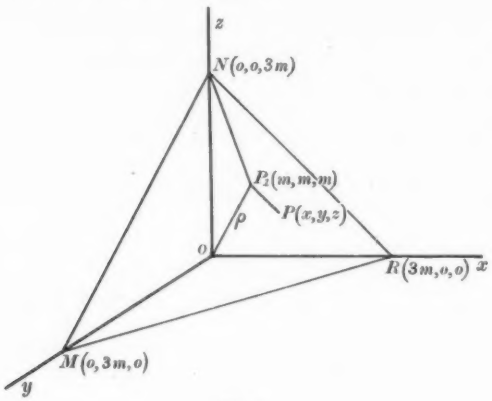


Fig. 1.

Equation (3) is a sphere with its centre at P_1 and its radius $s\sqrt{3}=P_1P$.

Let $\rho=m\sqrt{3}=OP_1$, $s'=s\sqrt{3}=P_1P$, $\theta=$angle P_1P makes with P_1N. Change to (ρ, s', θ) as a new co-ordinate system. Note that (s', θ) may be regarded as the polar co-ordinates of P in the plane MNR with the pole at P_1 and polar axis P_1N. Then the probability of a set of variates x, y, z falling in the element of $s'\,d\rho\,ds'\,d\theta$ is $\frac{1}{a^3}s'\,d\rho\,ds'\,d\theta$ to within the infinitesimals of higher order. Our problem will be solved if we can find the probability that in a given trial the value of s' will fall into ds'. This probability is equal to the double integral

* *Biometrika*, Vol. xxi (1929), pp. 124—143.

(4) $\frac{1}{a^3}\iiint s'\,d\rho\,d\theta\,ds'$ with respect to ρ and θ throughout the region of a cube of edge a (Fig. 2), in which

$$0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant a, \quad 0 \leqslant z \leqslant a,$$

and for which s' has an assigned value in the interval ds'.

This region of integration is the portion of a circular cylindrical surface* with an axis OD and a radius s' included within the cube of edge a (Fig. 2).

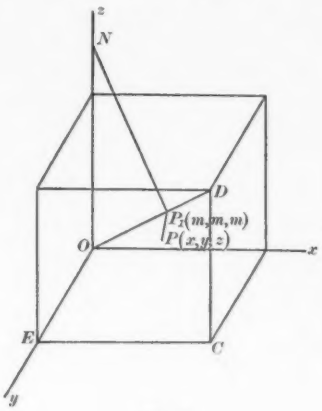


Fig. 2.

In the solution of a problem proposed by Rider, *American Mathematical Monthly*, Vol. XXXVIII (1931), pp. 292—5, the writer gave the results of this integration for $a=1$. When extended in an obvious manner to drawings from the interval 0 to a, instead of 0 to 1, we obtain

(5) $$\frac{6}{a^3}\left[\tfrac{1}{3}\pi a\sqrt{3}-s'\sqrt{6}\right]s'\,ds' \text{ for } 0 \leqslant s' \leqslant \tfrac{1}{2}a\sqrt{2},$$

(6) $$\frac{6}{a^3}\left\{-a\sqrt{3}\arccos\left(\frac{a^2}{s'^2}-1\right)+2\sqrt{6s'^2-3a^2}+\tfrac{1}{3}\pi a\sqrt{3}-s'\sqrt{6}\right\}s'\,ds' \text{ for } \tfrac{1}{2}a\sqrt{2} \leqslant s' \leqslant \tfrac{1}{3}a\sqrt{6}.$$

Remembering that $s'=s\sqrt{3}$, where s is the standard deviation,

$$ds'=\sqrt{3}ds.$$

Then by substitution we obtain from (5) and (6),

$$\frac{6\sqrt{3}}{a^3}\left[\pi a-3s\sqrt{6}\right]s\,ds \text{ for } 0 \leqslant s \leqslant \tfrac{1}{6}a\sqrt{6},$$

$$\frac{18}{a^3}\left\{-a\sqrt{3}\arccos\left(\frac{a^2}{3s^2}-1\right)+2\sqrt{18s^2-3a^2}+\tfrac{1}{3}\pi a\sqrt{3}-3s\sqrt{2}\right\}s\,ds \text{ for } \tfrac{1}{6}a\sqrt{6} \leqslant s \leqslant \tfrac{1}{3}a\sqrt{2}.$$

Thus, the frequency function of s is given by

(7) $$f(s)=\frac{6\sqrt{3}}{a^3}\left[\pi a-3s\sqrt{6}\right]s \text{ when } 0 \leqslant s \leqslant \tfrac{1}{6}a\sqrt{6},$$

and

(8) $$f(s)=\frac{18}{a^3}\left\{-a\sqrt{3}\arccos\left(\frac{a^2}{3s^2}-1\right)+2\sqrt{18s^2-3a^2}+\tfrac{1}{3}\pi a\sqrt{3}-3s\sqrt{2}\right\}s,$$

when $$\tfrac{1}{6}a\sqrt{6} \leqslant s \leqslant \tfrac{1}{3}a\sqrt{2}.$$

* Cf. E. S. Pearson, *Biometrika*, Vol. XIX (1927), pp. 244—246.

We find the derivative

$$f'(s)=\frac{6\sqrt{3}}{a^3}(\pi a-6s\sqrt{6}), \quad 0\leqslant s\leqslant \tfrac{1}{6}a\sqrt{6}, \tag{9}$$

$$=\frac{18}{a^3}\left[-a\sqrt{3}\arccos\left(\frac{a^2}{3s^2}-1\right)+4\sqrt{18s^2-3a^2}+\tfrac{1}{3}\pi a\sqrt{3}-6s\sqrt{2}\right], \tag{10}$$

$$s\leqslant \tfrac{1}{6}a\sqrt{6}\leqslant \tfrac{1}{3}a\sqrt{2}.$$

By substituting $s=\frac{1}{6}a\sqrt{6}$ in (7), (8), (9) and (10), we note that there is continuity both in the two curves and in their slopes at the junction at $s=\frac{1}{6}a\sqrt{6}$. Further, at the upper bound, $s=\frac{1}{3}a\sqrt{2}$, of the distributions, $f(s)=f'(s)=0$. A sketch of the curve is shown in Fig. 3 for $a=4$. We may note that the distribution curve for the interval 0 to $\frac{1}{6}a\sqrt{6}$ on s is a parabola, but the

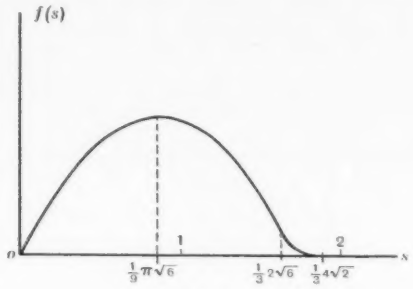


Fig. 3.

distribution curve for the interval $\frac{1}{6}a\sqrt{6}$ to $\frac{1}{3}a\sqrt{2}$ on s is a transcendental curve with a rather complicated equation. We may note further that the interval 0 to $\frac{1}{6}a\sqrt{6}$ is more than five-sixths of the total interval 0 to $\frac{1}{3}a\sqrt{2}$. Thus, the distribution of s from samples of three items is a parabola for more than five-sixths of the total range on s. Again, we find when $a=4$ that 98·86 % of the area is under the parabola to the left of the ordinate $s=\frac{1}{6}a\sqrt{6}$ at the junction with the more complicated curve.

Since the distribution of s from two items is a straight line, and from three items is a parabola for the major portion of the range, it is natural to surmise that the distribution of s from n items may be a polynomial of degree $n-1$ for an interval of the range of s near zero, but attempts to prove or disprove this conjecture seem to lead to difficult problems of n dimensional geometry.

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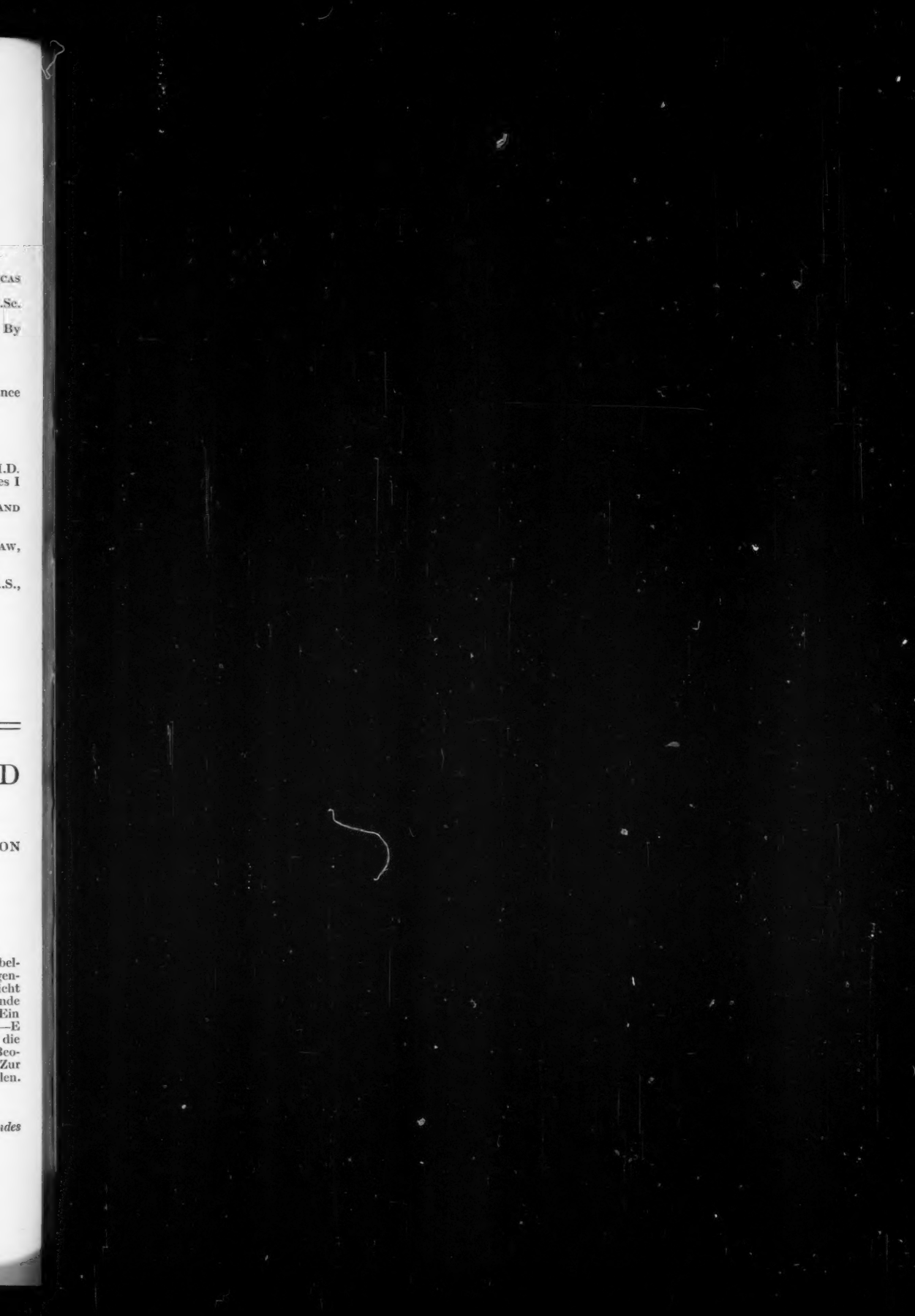

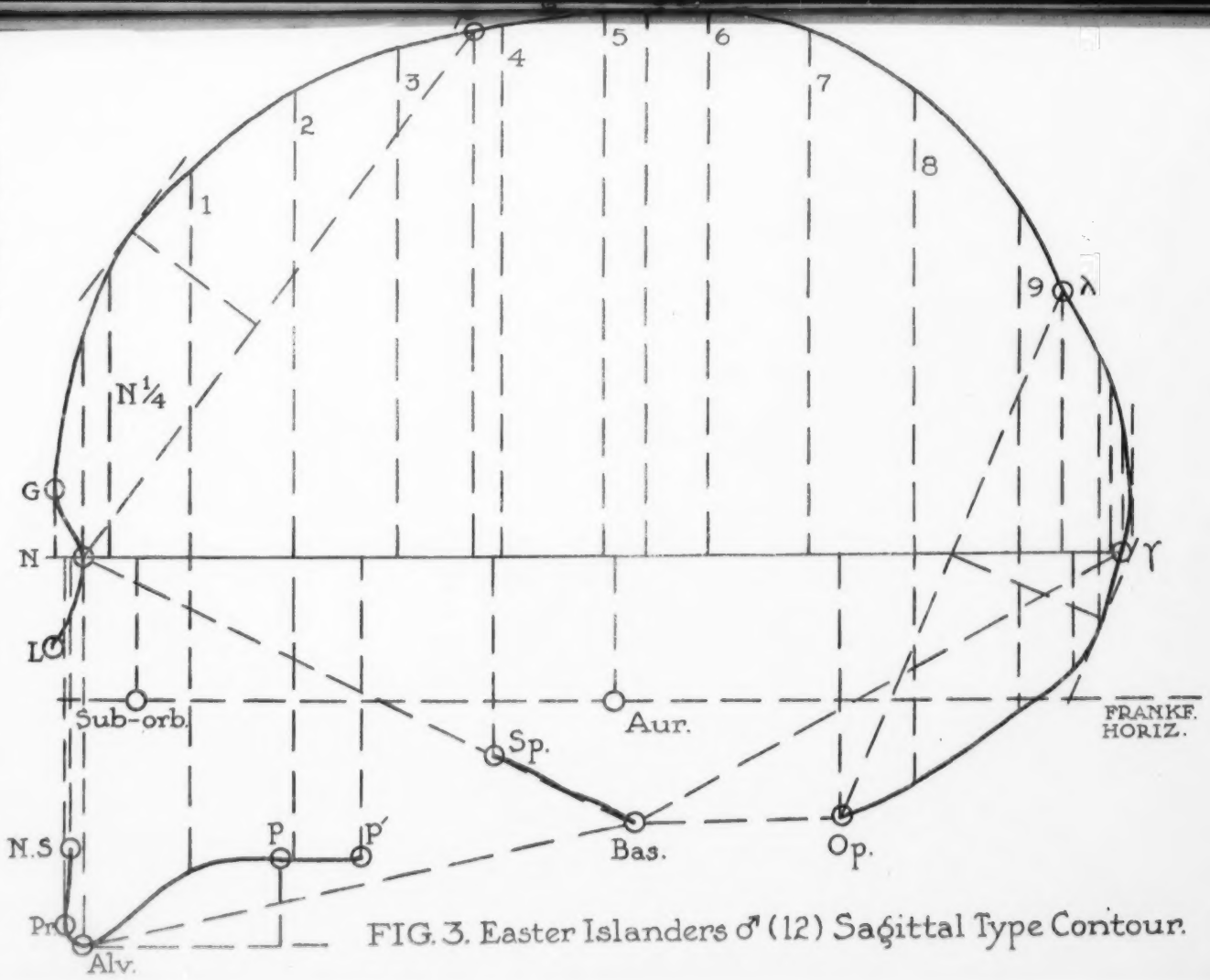

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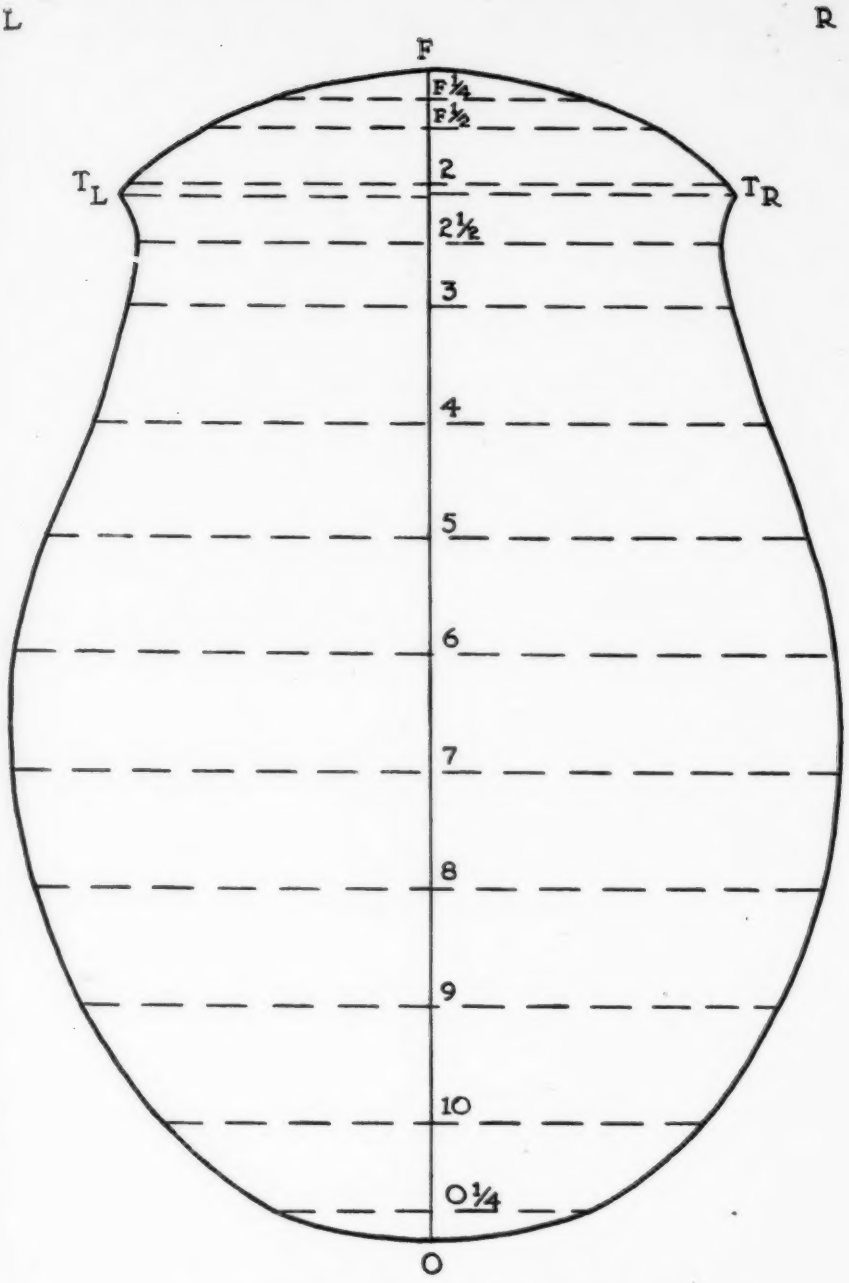

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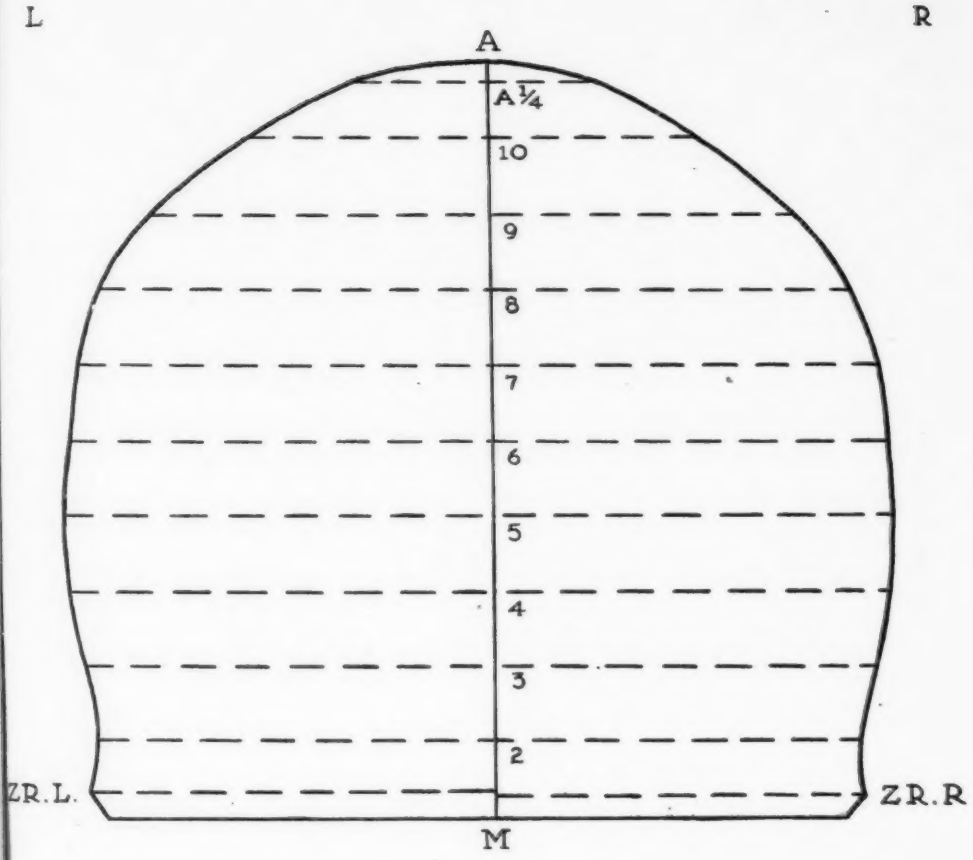

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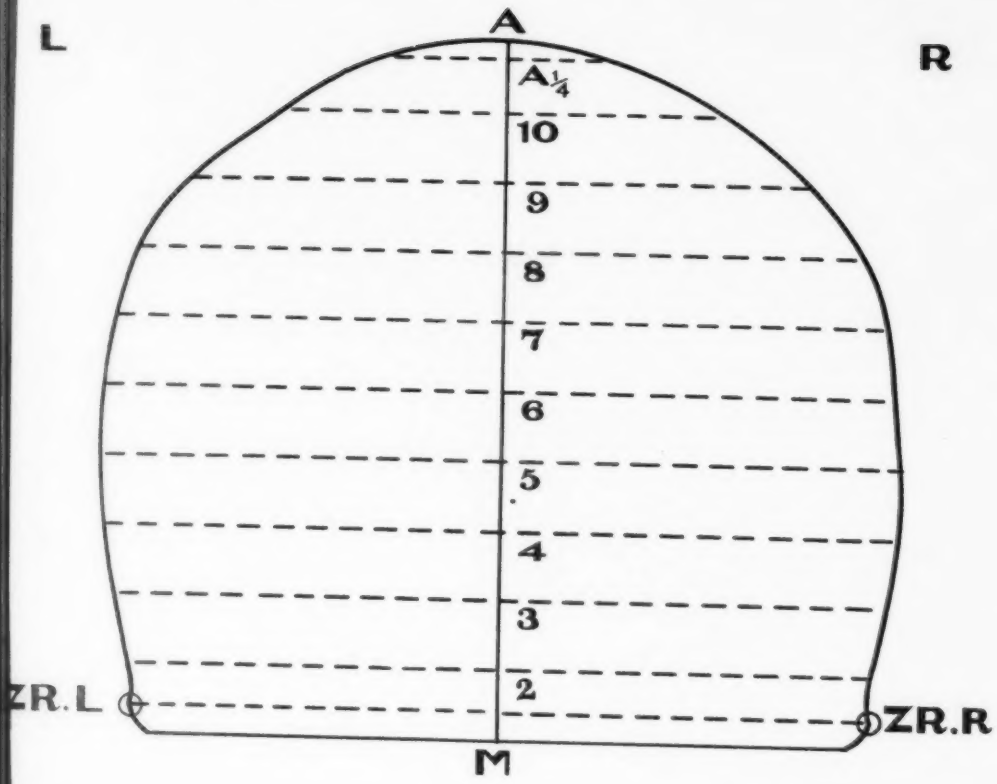
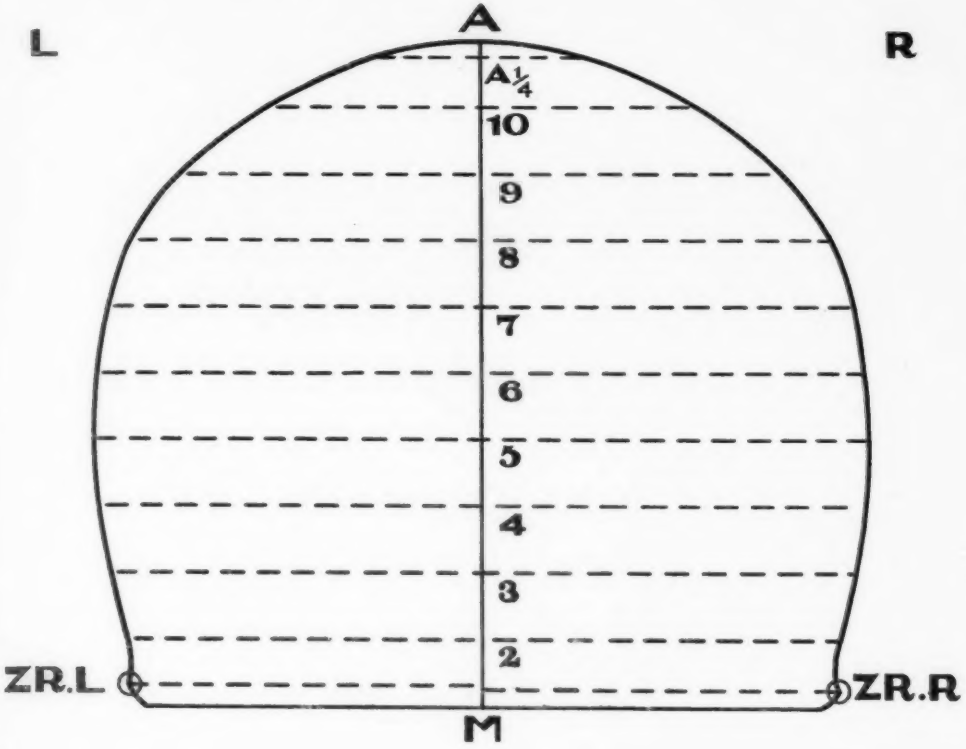


FIG. 3. Transverse Type Contour based on 48 ♂ Teita Skulls.



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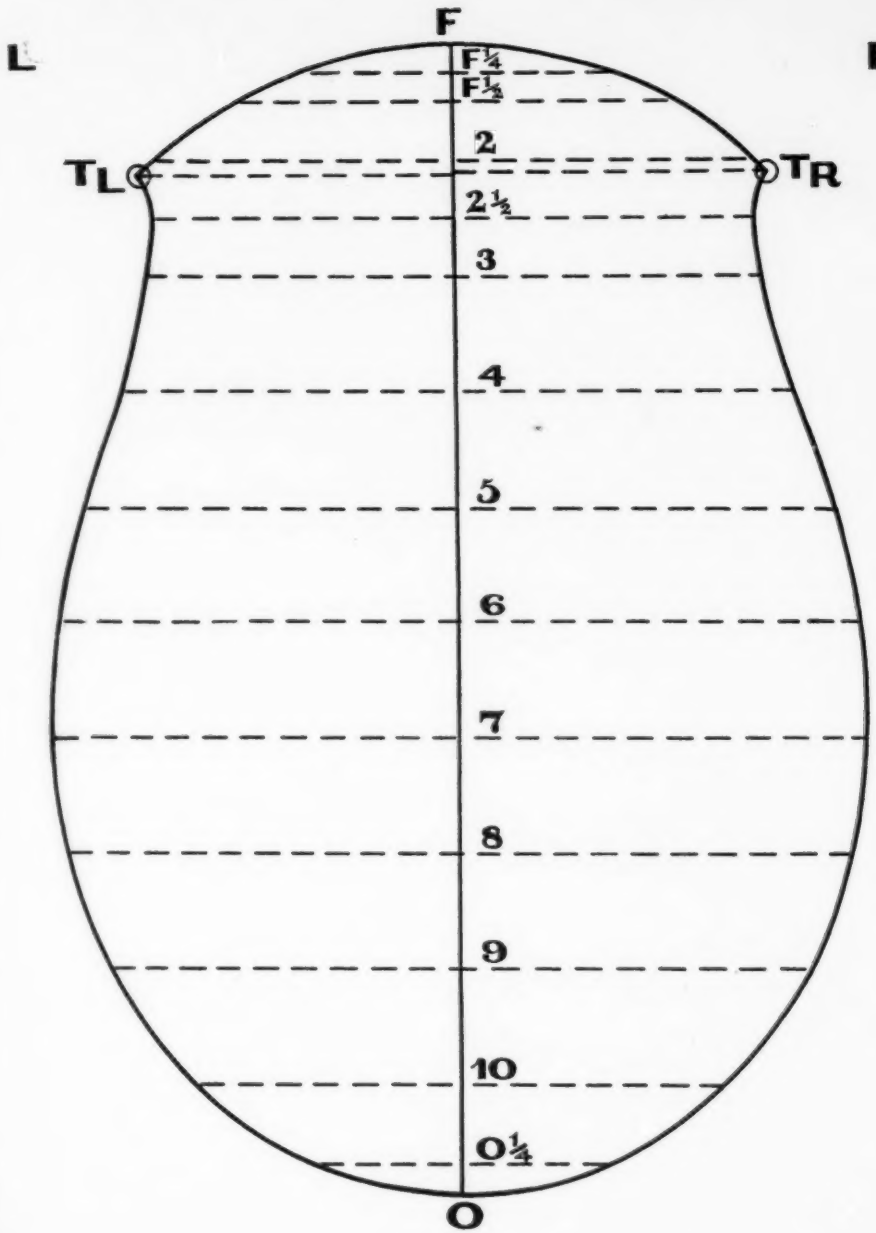

FIG. 5. Horizontal Type Contour based on 46 ♂ Teita Skulls.

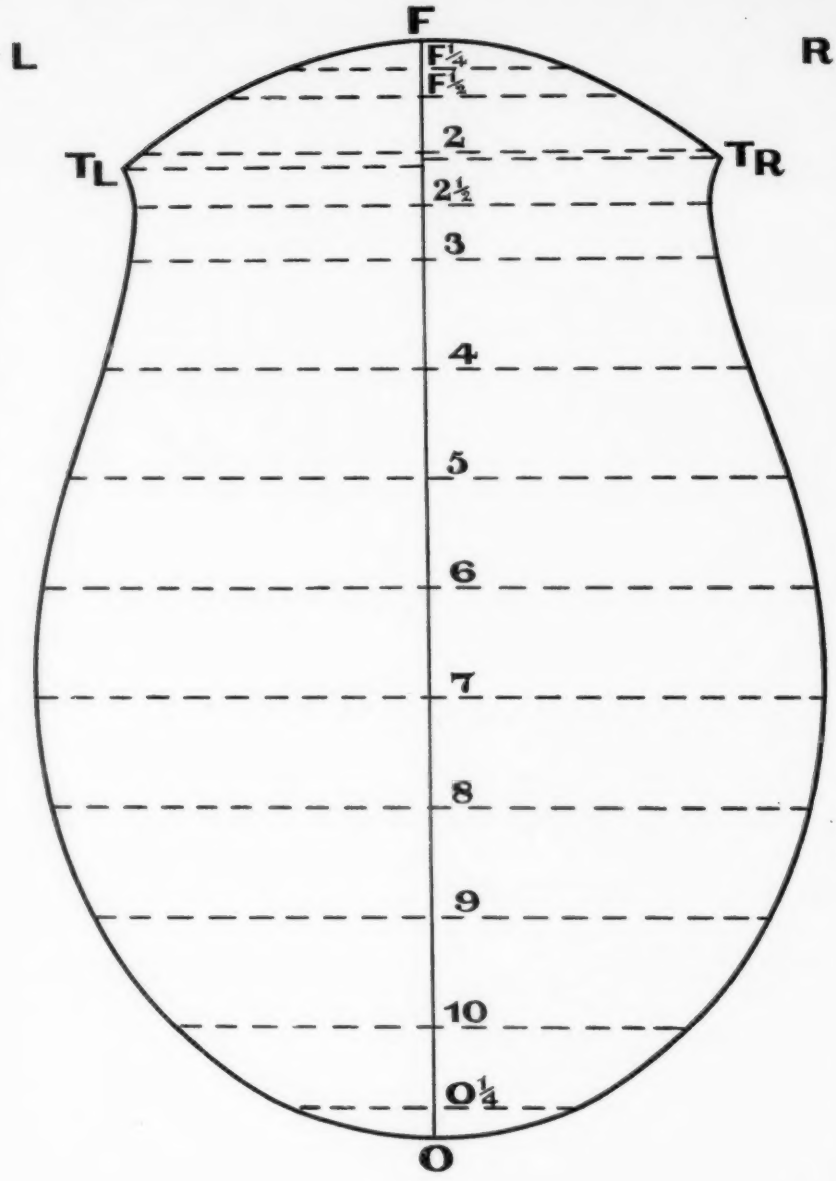

FIG. 6. Horizontal Type Contour based on 49 ♀ Teita Skulls.

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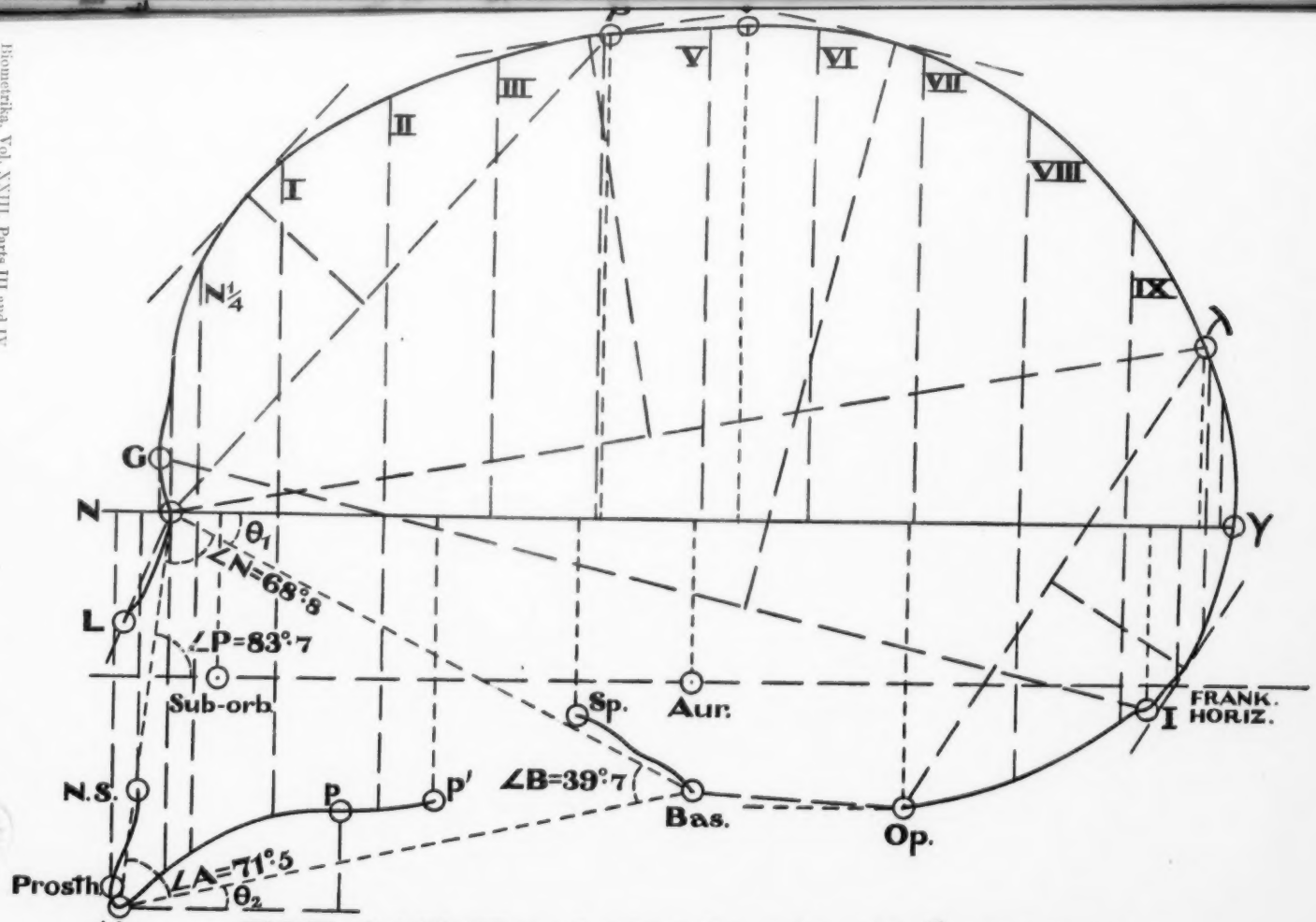

FIG. 7. Sagittal Type Contour based on 46 ♂ Teita Skulls.

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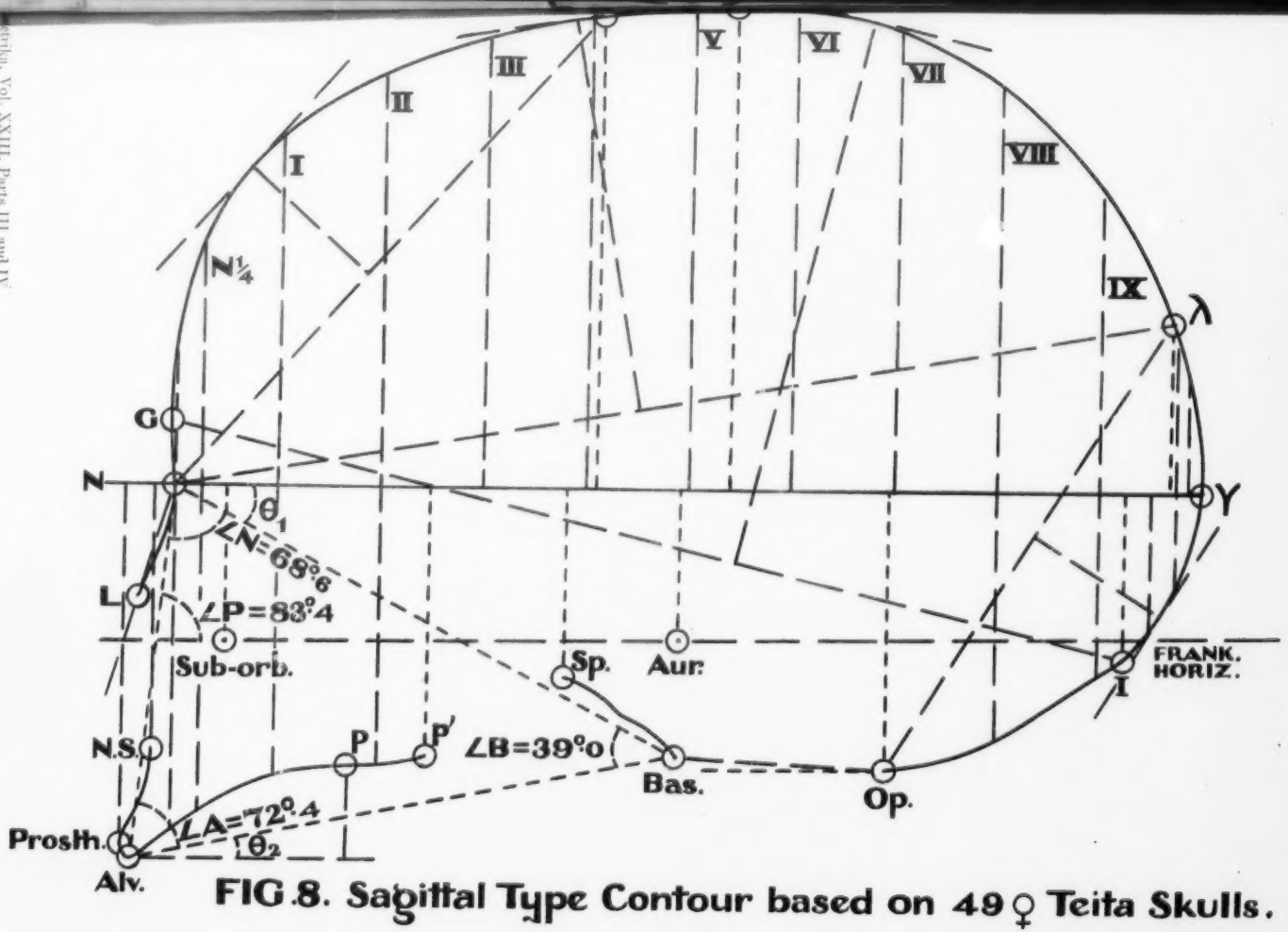

FIG. 8. Sagittal Type Contour based on 49 ♀ Teita Skulls.

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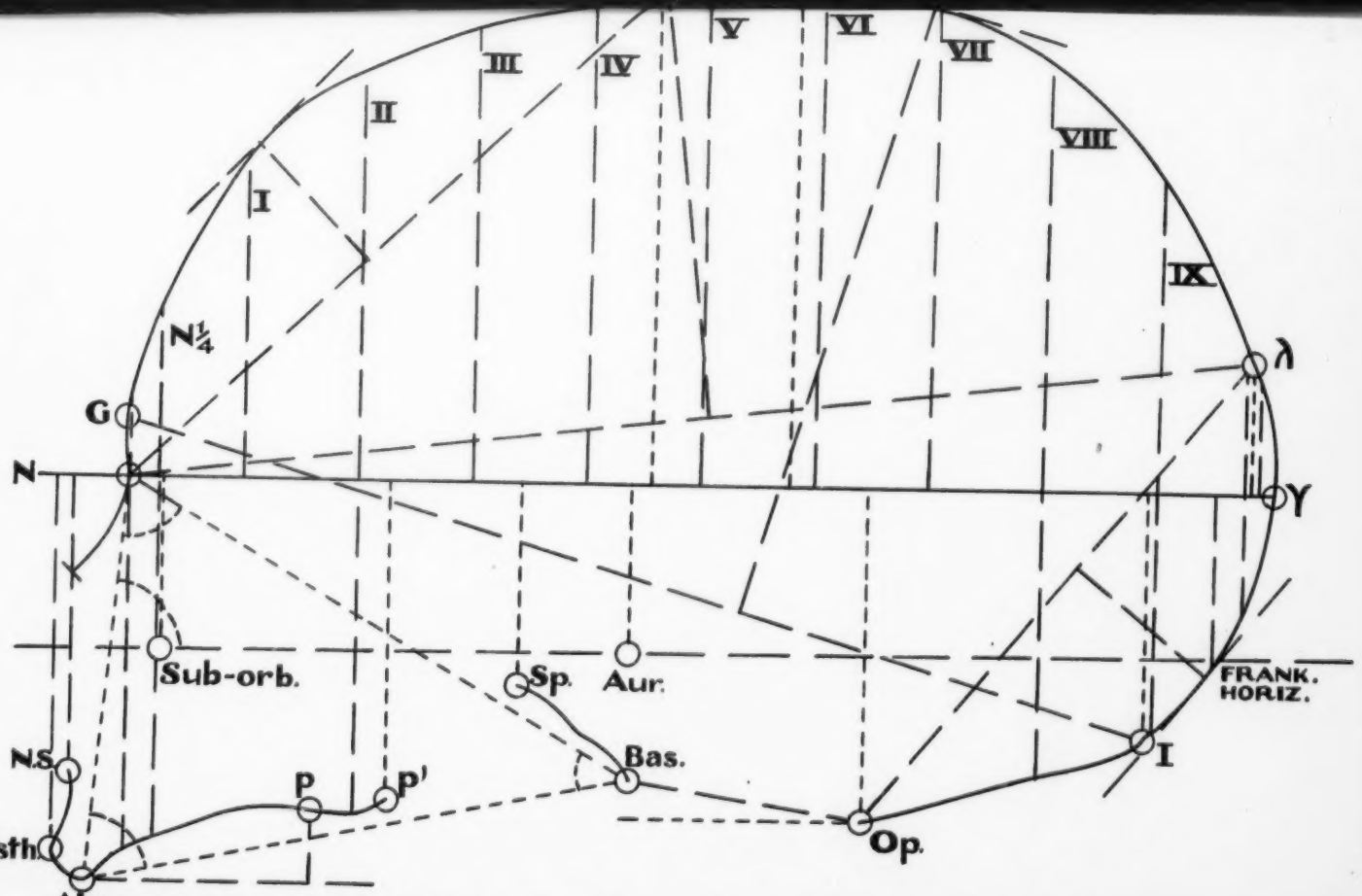

FIG. 9. Sagittal Type Contour based on 6♂ Naishi River Skulls.

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